

CHSE Odisha · +2 Science

MATHEMATICS

+2 2nd Year (Class XII) · As per CHSE Syllabus 2023-24

Theory in simple language · Formula boxes
Solved problems · Exam Q&A · Practice & revision

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MATHEMATICS

+2 2nd Year (Class XII) · Theory 80 + Internal 20

Exam Pattern & How to Study

Mathematics - Unit Weightage

Class XI Unit	Marks	Class XII Unit	Marks
Sets and Functions	23	Relations and Functions	08
Algebra	25	Algebra (Matrices, Determinants)	10
Coordinate Geometry	12	Calculus	35
Calculus	08	Vectors and 3-D Geometry	14
Statistics and Probability	12	Linear Programming	05
		Probability	08

Question typology: about 55% Remembering/Understanding, 25% Applying, 20% Analysing/Evaluating/Creating. There is no overall choice, but about 33% internal choice in every section.

GOLDEN RULES FOR WRITING ANSWERS

- Always write the **formula first**, then substitute values, then the answer **with units**.
- Underline or box the final answer. Examiners mark step-by-step, so even a wrong final answer earns step marks.
- For theory answers, use short points and draw a diagram wherever possible.
- Attempt questions you know best first, but keep their numbering exactly as in the paper.

XII-1. Relations and Functions

Recap: A relation R from set A to A is a subset of $A \times A$. We now study special types of relations and functions.

Types of relations

Type	Condition	Example on $A = \{1, 2, 3\}$
Empty	$R = \varnothing$	" $x - y = 10$ "
Universal	$R = A \times A$	" $ x - y \geq 0$ "
Reflexive	$(a, a) \in R$ for every $a \in A$	must contain $(1,1), (2,2), (3,3)$
Symmetric	$(a, b) \in R \Rightarrow (b, a) \in R$	if $(1,2)$ then $(2,1)$
Transitive	$(a, b), (b, c) \in R \Rightarrow (a, c) \in R$	if $(1,2), (2,3)$ then $(1,3)$
Equivalence	Reflexive + Symmetric + Transitive	"is equal to", "is parallel to", congruence

Equivalence class $[a]$: set of all elements related to a . Equivalence classes divide the set into disjoint parts.

Types of functions

ONE-ONE, ONTO, BIJECTIVE

- **One-one (injective):** different inputs give different outputs. Test: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. Graph test: any horizontal line cuts the graph at most once.
- **Many-one:** two or more inputs have the same output (e.g. $f(x) = x^2$ on $\mathbb{R}$: $f(2) = f(-2)$).
- **Onto (surjective):** every element of the co-domain is an image, i.e. range = co-domain. Test: for every y , solve $y = f(x)$ and check x is in the domain.
- **Bijective:** both one-one and onto. Only bijective functions have an inverse.

Example 1. Show that $R = \{(a, b) : 2 \text{ divides } (a - b)\}$ on $\mathbb{Z}$ is an equivalence relation.

Reflexive: $a - a = 0$, divisible by 2 ✓. **Symmetric:** if $2 \mid (a - b)$ then $2 \mid (b - a) = -(a - b)$ ✓. **Transitive:** if $2 \mid (a - b)$ and $2 \mid (b - c)$, then $a - c = (a - b) + (b - c)$ is divisible by 2 ✓. Hence equivalence. Classes: $[0] = \text{even integers}$, $[1] = \text{odd integers}$.

Example 2. Check whether $R = \{(1,1), (2,2), (3,3), (1,2), (2,3)\}$ on $\{1,2,3\}$ is reflexive, symmetric, transitive.

Reflexive ✓ (all (a,a) present). Not symmetric: $(1,2) \in R$ but $(2,1) \notin R$. Not transitive: $(1,2), (2,3) \in R$ but $(1,3) \notin R$.

Example 3. Show that $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 3 - 4x$ is bijective.

One-one: $3 - 4x_1 = 3 - 4x_2 \Rightarrow x_1 = x_2$. **Onto:** for any y , $x = (3 - y)/4$ is real and $f(x) = y$. So f is bijective.

Example 4. Is $f : \mathbf{N} \rightarrow \mathbf{N}$, $f(x) = x^2$ one-one? onto?

One-one ✓ (on natural numbers $x^2 = y^2 \Rightarrow x = y$). Not onto: 2 has no pre-image in $\mathbf{N}$.

Example 5. Show that $f : \mathbf{R} \rightarrow \mathbf{R}$, $f(x) = x^2$ is neither one-one nor onto.

$f(1) = f(-1) = 1 \Rightarrow$ not one-one. -1 in co-domain has no pre-image $\Rightarrow$ not onto.

XII-2. Inverse Trigonometric Functions

Idea: $\sin x$ is not one-one on $\mathbf{R}$ (it repeats). If we restrict its domain to $[-\pi/2, \pi/2]$, it becomes one-one and onto $[-1, 1]$, so it has an inverse, written $\sin^{-1}x$ or $\arcsin x$. $\sin^{-1}x$ means "the angle whose sine is x ". It is **not** $1/\sin x$.

Function	Domain	Range (principal value branch)
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	$\mathbf{R}$	$(-\pi/2, \pi/2)$
$\cot^{-1}x$	$\mathbf{R}$	$(0, \pi)$
$\sec^{-1}x$	$\mathbf{R} - (-1, 1)$	$[0, \pi] - \{\pi/2\}$
$\operatorname{cosec}^{-1}x$	$\mathbf{R} - (-1, 1)$	$[-\pi/2, \pi/2] - \{0\}$

Graphs: the graph of $\sin^{-1}x$ is obtained by reflecting the graph of $\sin x$ (on $[-\pi/2, \pi/2]$) in the line $y = x$. Similarly for the others.

USEFUL PROPERTIES

$$\sin^{-1}(-x) = -\sin^{-1}x \quad \tan^{-1}(-x) = -\tan^{-1}x \quad \cos^{-1}(-x) = \pi - \cos^{-1}x$$

$$\sin^{-1}x + \cos^{-1}x = \pi/2 \quad \tan^{-1}x + \cot^{-1}x = \pi/2$$

$$\sin^{-1}(1/x) = \operatorname{cosec}^{-1}x \quad \cos^{-1}(1/x) = \sec^{-1}x$$

Example 1. Find the principal value of $\sin^{-1}(1/2)$, $\cos^{-1}(-1/2)$, $\tan^{-1}(-\sqrt{3})$.

$$\sin^{-1}(1/2) = \pi/6. \quad \cos^{-1}(-1/2) = \pi - \pi/3 = 2\pi/3. \quad \tan^{-1}(-\sqrt{3}) = -\pi/3.$$

Example 2. Find the value of $\sin^{-1}(\sin 3\pi/5)$.

$3\pi/5$ is not in $[-\pi/2, \pi/2]$. $\sin(3\pi/5) = \sin(\pi - 3\pi/5) = \sin(2\pi/5)$, and $2\pi/5$ is in range. Answer $2\pi/5$.

Example 3. Evaluate $\cos^{-1}(1/2) + 2 \sin^{-1}(1/2)$.

$$\pi/3 + 2(\pi/6) = 2\pi/3.$$

Example 4. Find the value of $\tan(\sin^{-1}(3/5))$.

$$\text{Let } \sin \theta = 3/5 \Rightarrow \cos \theta = 4/5 \Rightarrow \tan \theta = 3/4.$$

XII-3. Matrices

Idea: A matrix is a rectangular arrangement of numbers in rows and columns, like a table of marks. Matrices let us handle many numbers and equations together.

DEFINITIONS

- **Order $m \times n$:** m rows, n columns. Element in i -th row, j -th column: a_{ij} .
- **Equal matrices:** same order and all corresponding elements equal.
- **Row matrix** ($1 \times n$), **column matrix** ($m \times 1$), **square matrix** ($m = n$).
- **Diagonal matrix:** square, all non-diagonal elements zero. **Scalar matrix:** diagonal with equal diagonal elements. **Identity matrix I :** diagonal elements 1, others 0.
- **Zero matrix O :** all elements zero.
- **Transpose A' (or A^T):** rows become columns.
- **Symmetric:** $A' = A$. **Skew-symmetric:** $A' = -A$ (diagonal elements are all 0).

Operations

RULES

Addition: only for same order; add corresponding elements. $A + B = B + A$; $(A + B) + C = A + (B + C)$

Scalar multiplication: kA multiplies every element by k

Multiplication AB : possible only if (columns of A) = (rows of B). If A is $m \times n$ and B is $n \times p$, AB is $m \times p$.

$(AB)_{ij}$ = sum of (i -th row of A) $\times$ (j -th column of B)

$AB \neq BA$ in general (not commutative); $A(BC) = (AB)C$; $A(B + C) = AB + AC$; $AI = IA = A$

$(A')' = A$; $(A + B)' = A' + B'$; $(kA)' = kA'$; $(AB)' = B'A'$

Every square matrix: $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$ = symmetric + skew-symmetric

SURPRISING FACTS

(1) AB can be zero even if $A \neq O$ and $B \neq O$. Example: $A = [0 \ 1; 0 \ 0]$, $B = [1 \ 0; 0 \ 0]$ gives $AB = [0 \ 0; 0 \ 0]$. (2) $AB = AC$ does not imply $B = C$.

Invertible matrix: A square matrix A is invertible if there is B such that $AB = BA = I$; $B = A^{-1}$. **The inverse is unique:** if B and C are both inverses, then $B = BI = B(AC) = (BA)C = IC = C$. Also $(AB)^{-1} = B^{-1}A^{-1}$.

Example 1. Construct a 2×2 matrix with $a_{ij} = (i + j)^2/2$.

$a_{11} = 2$, $a_{12} = 9/2$, $a_{21} = 9/2$, $a_{22} = 8$. $A = \begin{bmatrix} 2 & 9/2 \\ 9/2 & 8 \end{bmatrix}$.

Example 2. If $A = [1 \ 2; 3 \ 4]$ and $B = [2 \ 0; 1 \ 3]$, find AB and BA .

$AB = [1 \cdot 2 + 2 \cdot 1, 1 \cdot 0 + 2 \cdot 3; 3 \cdot 2 + 4 \cdot 1, 3 \cdot 0 + 4 \cdot 3] = [4 \ 6; 10 \ 12]$. $BA = [2 \cdot 1 + 0 \cdot 3, 2 \cdot 2 + 0 \cdot 4; 1 \cdot 1 + 3 \cdot 3, 1 \cdot 2 + 3 \cdot 4] = [2 \ 4; 10 \ 14]$. So $AB \neq BA$.

Example 3. Find x and y if $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$.

$2 + y = 5 \Rightarrow y = 3$; $2x + 2 = 8 \Rightarrow x = 3$. **$x = 3, y = 3$.**

Example 4. Express $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

$A' = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$. $P = \frac{1}{2}(A + A') = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix}$ (symmetric). $Q = \frac{1}{2}(A - A') = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ (skew). $A = P + Q$.

Example 5. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find k so that $A^2 = kA - 2I$.

$A^2 = \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$. $kA - 2I = \begin{bmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{bmatrix}$.
Compare: $-2k = -2 \Rightarrow k = 1$ (check: $3-2 = 1$ ✓).

XII-4. Determinants

Idea: Every square matrix has a number attached to it called its determinant, $|A|$ or $\det A$. It tells us whether the matrix has an inverse ($|A| \neq 0$) and helps solve equations and find areas.

FORMULAS

2×2 : $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3×3 (expand along R_1): $a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

Minor M_{ij} = determinant left after deleting row i and column j

Cofactor $A_{ij} = (-1)^{i+j}M_{ij}$

$|A|$ = sum of (elements of any row/column $\times$ their cofactors)

$\text{adj } A$ = transpose of the cofactor matrix $A(\text{adj } A) = (\text{adj } A)A = |A| I$

$A^{-1} = (1/|A|) \text{adj } A$ (exists only if $|A| \neq 0$, i.e. A is non-singular)

$|AB| = |A||B|$ $|A'| = |A|$ $|kA| = k^n|A|$ (n = order) $|\text{adj } A| = |A|^{n-1}$

Area of triangle = $\frac{1}{2} |x_1 y_1 1; x_2 y_2 1; x_3 y_3 1|$ (take absolute value). Area = 0 $\Rightarrow$ points collinear.

For 2×2 $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$: $\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Solving linear equations by matrix method

Write the system as $AX = B$. Then:

- If $|A| \neq 0$: unique solution $X = A^{-1}B$ (consistent).
- If $|A| = 0$ and $(\text{adj } A)B \neq O$: no solution (inconsistent).
- If $|A| = 0$ and $(\text{adj } A)B = O$: infinitely many solutions or none (check further).

Example 1. Evaluate $|\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \\ 4 & 1 & -1 \end{bmatrix}|$.

$= 2(0 \cdot (-1) - 2 \cdot 1) - (-1)(1 \cdot (-1) - 2 \cdot 4) + 3(1 \cdot 1 - 0 \cdot 4) = 2(-2) + 1(-9) + 3(1) = -4 - 9 + 3 = -10$.

Example 2. Find the area of the triangle with vertices (1, 0), (6, 0), (4, 3).

$$\frac{1}{2}|1(0 - 3) - 0 + 1(18 - 0)| = \frac{1}{2}|-3 + 18| = 7.5 \text{ sq units.}$$

Example 3. Find k if the area of the triangle with vertices (k, 0), (4, 0), (0, 2) is 4 sq units.

$$\frac{1}{2}|k(0 - 2) - 0 + 1(8 - 0)| = 4 \Rightarrow |8 - 2k| = 8 \Rightarrow k = 0 \text{ or } k = 8.$$

Example 4. Find the inverse of $A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$.

$$|A| = 6 + 8 = 14. \text{ adj } A = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}. A^{-1} = (1/14)\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}.$$

Example 5. Find A^{-1} for $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$.

$$|A| = 1(10 - 0) - 2(0) + 3(0) = 10. \text{ Cofactors: } A_{11} = 10, A_{12} = 0, A_{13} = 0; A_{21} = -(10 - 0) = -10, A_{22} = 5, A_{23} = 0; A_{31} = (8 - 6) = 2, A_{32} = -(4 - 0) = -4, A_{33} = 2. \\ \text{adj } A = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}. A^{-1} = (1/10)\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}.$$

Example 6. Solve by matrix method: $2x + 5y = 1, 3x + 2y = 7$.

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}. |A| = 4 - 15 = -11. \text{ adj } A = \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}. X = A^{-1}B = (-1/11)\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix} = (-1/11)\begin{bmatrix} -33 & 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}. \mathbf{x = 3, y = -1.}$$

Example 7. Solve: $x + y + z = 6, y + 3z = 11, x - 2y + z = 0$.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}, |A| = 1(1 + 6) - 1(0 - 3) + 1(0 - 1) = 7 + 3 - 1 = 9. \\ \text{Cofactors: } A_{11} = 7, A_{12} = 3, A_{13} = -1; A_{21} = -(1 + 2) = -3, A_{22} = 0, A_{23} = -(-2 - 1) = 3; A_{31} = 3 - 1 = 2, A_{32} = -(3 - 0) = -3, A_{33} = 1. \\ \text{adj } A = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}. X = (1/9)\begin{bmatrix} 42 - 33 + 0 \\ 18 + 0 - 0 \\ -6 + 33 + 0 \end{bmatrix} = (1/9)\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix} = \mathbf{x = 1, y = 2, z = 3.}$$

Example 8. Check consistency: $x + 3y = 5, 2x + 6y = 8$.

$$|A| = 6 - 6 = 0. \text{ adj } A = \begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}. (\text{adj } A)B = \begin{bmatrix} 30 - 24 \\ -10 + 8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \neq O \Rightarrow \mathbf{\text{inconsistent (no solution).}}$$

Exam Q&A

Q1. If A is a 3×3 matrix with $|A| = 4$, find $|2A|$ and $|\text{adj } A|$. [2]

$$|2A| = 2^3 \times 4 = 32. |\text{adj } A| = 4^2 = 16.$$

Q2. What is the principal value of $\cos^{-1}(\cos 7\pi/6)$? [1]

$$\cos(7\pi/6) = \cos(5\pi/6), \text{ and } 5\pi/6 \in [0, \pi]. \text{ Answer } 5\pi/6.$$

Q3. If A is skew-symmetric of order 3, what is $|A|$? [1]

0 (determinant of an odd-order skew-symmetric matrix is zero).

Q4. Show that the relation "is congruent to" on the set of triangles is an equivalence relation. [4]

Every triangle is congruent to itself (reflexive); if $T_1 \cong T_2$ then $T_2 \cong T_1$ (symmetric); if $T_1 \cong T_2$ and $T_2 \cong T_3$ then $T_1 \cong T_3$ (transitive).

XII-5. Continuity and Differentiability

Continuity

Idea: A function is continuous at a point if you can draw its graph through that point without lifting your pen — no break, jump or hole.

DEFINITION

f is continuous at $x = a$ if: (1) $f(a)$ is defined, (2) $\lim_{x \rightarrow a} f(x)$ exists (LHL = RHL), and (3) $\lim_{x \rightarrow a} f(x) = f(a)$. In short: **LHL = RHL = $f(a)$** .

Polynomials, $\sin x$, $\cos x$, e^x , $|x|$ are continuous everywhere. $\log x$ is continuous for $x > 0$; $\tan x$ except at odd multiples of $\pi/2$; rational functions except where the denominator is zero. Sums, differences, products, quotients (denominator $\neq 0$) and compositions of continuous functions are continuous. $[x]$ is discontinuous at every integer.

Example 1. Check continuity of $f(x) = 2x + 3$ ($x \leq 2$), $2x - 3$ ($x > 2$) at $x = 2$.

LHL = 7, RHL = 1. LHL $\neq$ RHL $\Rightarrow$ **discontinuous at $x = 2$** .

Example 2. Find k so that $f(x) = kx + 1$ ($x \leq 5$), $3x - 5$ ($x > 5$) is continuous at $x = 5$.

$5k + 1 = 15 - 5 = 10 \Rightarrow$ **$k = 9/5$** .

Example 3. Find k if $f(x) = (\sin 3x)/x$ for $x \neq 0$ and $f(0) = k$ is continuous at 0.

$\lim_{x \rightarrow 0} \sin 3x/x = 3$. So **$k = 3$** .

Differentiability

f is differentiable at a if the left-hand derivative equals the right-hand derivative. **Every differentiable function is continuous, but a continuous function need not be differentiable.** Example: $|x|$ is continuous at 0 but not differentiable there (sharp corner: LHD = -1 , RHD = $+1$).

DERIVATIVES TO MEMORISE

$$d/dx(x^n) = nx^{n-1} \quad d/dx(e^x) = e^x \quad d/dx(a^x) = a^x \log a \quad d/dx(\log x) = 1/x$$

$$d/dx \sin x = \cos x \quad \cos x \rightarrow -\sin x \quad \tan x \rightarrow \sec^2 x \quad \cot x \rightarrow -\operatorname{cosec}^2 x \quad \sec x \rightarrow \sec x \tan x \\ \operatorname{cosec} x \rightarrow -\operatorname{cosec} x \cot x$$

$$d/dx \sin^{-1} x = 1/\sqrt{1-x^2} \quad \cos^{-1} x \rightarrow -1/\sqrt{1-x^2} \quad \tan^{-1} x \rightarrow 1/(1+x^2) \\ \cot^{-1} x \rightarrow -1/(1+x^2) \quad \sec^{-1} x \rightarrow 1/(|x|\sqrt{x^2-1}) \quad \operatorname{cosec}^{-1} x \rightarrow -1/(|x|\sqrt{x^2-1})$$

Chain rule: if $y = f(u)$, $u = g(x)$: $dy/dx = (dy/du)(du/dx)$

Parametric: $x = f(t)$, $y = g(t)$: $dy/dx = (dy/dt)/(dx/dt)$

Second derivative: $d^2y/dx^2 = d/dx(dy/dx)$

Exponential and logarithmic functions (concept)

$e \approx 2.718$ is a special number. $y = e^x$ is always positive and increasing; its inverse is the natural logarithm $y = \log_e x = \ln x$ (defined for $x > 0$). Laws: $\log(mn) = \log m + \log n$; $\log(m/n) = \log m - \log n$; $\log m^n = n \log m$; $\log_a x = \log x / \log a$.

Types of differentiation problems

(a) Chain rule

Differentiate $y = \sin(x^2 + 5)$.

$$dy/dx = \cos(x^2 + 5) \cdot 2x = 2x \cos(x^2 + 5).$$

Differentiate $y = \sqrt{3x + 2} + 1/\sqrt{2x^2 + 4}$.

$$y' = 3/(2\sqrt{3x + 2}) + (-1/2)(2x^2 + 4)^{-3/2}(4x) = 3/(2\sqrt{3x + 2}) - 2x/(2x^2 + 4)^{3/2}.$$

(b) Implicit functions — differentiate both sides w.r.t. x , treating y as a function of x ($d/dx(y^2) = 2y dy/dx$).

Find dy/dx if $x^2 + xy + y^2 = 100$.

$$2x + (y + x y') + 2y y' = 0 \Rightarrow y'(x + 2y) = -(2x + y) \Rightarrow dy/dx = -(2x + y)/(x + 2y).$$

Find dy/dx if $\sin^2 y + \cos xy = \pi$.

$$2 \sin y \cos y y' - \sin(xy)(y + x y') = 0 \Rightarrow y'(\sin 2y - x \sin xy) = y \sin xy \Rightarrow dy/dx = y \sin xy / (\sin 2y - x \sin xy).$$

(c) Inverse trigonometric — use substitution

Differentiate $y = \tan^{-1}[2x/(1 - x^2)]$, $-1 < x < 1$.

$$\text{Put } x = \tan \theta: y = \tan^{-1}(\tan 2\theta) = 2\theta = 2 \tan^{-1} x \Rightarrow dy/dx = 2/(1 + x^2).$$

Differentiate $y = \sin^{-1}(2x\sqrt{1 - x^2})$, $-1/\sqrt{2} < x < 1/\sqrt{2}$.

$$\text{Put } x = \sin \theta: 2 \sin \theta \cos \theta = \sin 2\theta \Rightarrow y = 2\theta = 2 \sin^{-1} x \Rightarrow dy/dx = 2/\sqrt{1 - x^2}.$$

(d) Logarithmic differentiation — used for x^x , $(\sin x)^{\cos x}$, or long products/quotients. Take log of both sides first.

Differentiate $y = x^x$.

$$\log y = x \log x \Rightarrow (1/y)y' = \log x + 1 \Rightarrow dy/dx = x^x(1 + \log x).$$

Differentiate $y = (\sin x)^x + \sin^{-1}\sqrt{x}$.

$$\text{Let } u = (\sin x)^x: \log u = x \log \sin x \Rightarrow u' = (\sin x)^x(\log \sin x + x \cot x). v = \sin^{-1}\sqrt{x} \Rightarrow v' = [1/\sqrt{1 - x}] \cdot 1/(2\sqrt{x}). dy/dx = u' + v'.$$

Differentiate $y = \sqrt{[(x-1)(x-2)/((x-3)(x-4)(x-5))]}$.

$$\log y = \frac{1}{2}[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)]. \quad dy/dx = (y/2)[1/(x-1) + 1/(x-2) - 1/(x-3) - 1/(x-4) - 1/(x-5)].$$

(e) Parametric form

Find dy/dx if $x = a(\theta - \sin \theta)$, $y = a(1 + \cos \theta)$.

$$dx/d\theta = a(1 - \cos \theta), \quad dy/d\theta = -a \sin \theta. \quad dy/dx = -\sin \theta / (1 - \cos \theta) = -2 \sin(\theta/2) \cos(\theta/2) / (2 \sin^2(\theta/2)) = -\cot(\theta/2).$$

Find dy/dx if $x = at^2$, $y = 2at$.

$$dy/dx = 2a/(2at) = 1/t.$$

(f) Second order derivatives

If $y = A \sin x + B \cos x$, prove that $d^2y/dx^2 + y = 0$.

$$y' = A \cos x - B \sin x; \quad y'' = -A \sin x - B \cos x = -y. \quad \text{So } y'' + y = 0.$$

If $y = 3e^{2x} + 2e^{3x}$, prove $y'' - 5y' + 6y = 0$.

$$y' = 6e^{2x} + 6e^{3x}; \quad y'' = 12e^{2x} + 18e^{3x}. \quad y'' - 5y' + 6y = (12 - 30 + 18)e^{2x} + (18 - 30 + 12)e^{3x} = 0.$$

If $y = \sin^{-1}x$, show $(1 - x^2)y'' - xy' = 0$.

$$y' = (1 - x^2)^{-1/2} \Rightarrow \sqrt{1 - x^2} y' = 1. \quad \text{Differentiate: } \sqrt{1 - x^2} y'' - x y' / \sqrt{1 - x^2} = 0. \\ \text{Multiply by } \sqrt{1 - x^2}: (1 - x^2)y'' - xy' = 0.$$

XII-6. Applications of Derivatives

(1) Rate of change

dy/dx measures how fast y changes when x changes. If both depend on time t , $dy/dt = (dy/dx)(dx/dt)$.

Example 1. The radius of a circle increases at 0.7 cm/s. How fast is its circumference increasing?

$$C = 2\pi r \Rightarrow dC/dt = 2\pi dr/dt = 2\pi \times 0.7 = 1.4\pi \text{ cm/s.}$$

Example 2. A balloon's radius increases at 2 cm/s. Find the rate of increase of volume when $r = 10$ cm.

$$V = (4/3)\pi r^3 \Rightarrow dV/dt = 4\pi r^2(dr/dt) = 4\pi(100)(2) = 800\pi \text{ cm}^3/\text{s.}$$

Example 3. Total cost $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$. Find the marginal cost when 17 units are produced.

$MC = C'(x) = 0.021x^2 - 0.006x + 15$. At $x = 17$: $0.021(289) - 0.102 + 15 = 6.069 - 0.102 + 15 = \text{Rs } 20.967$.

(2) Increasing and decreasing functions

TEST

$f'(x) > 0$ on an interval $\Rightarrow f$ is strictly increasing; $f'(x) < 0 \Rightarrow$ strictly decreasing; $f'(x) = 0$ throughout $\Rightarrow$ constant.

Method: find $f'(x)$, solve $f'(x) = 0$ to get critical points, divide the number line, check sign of f' in each part.

Example 4. Find intervals where $f(x) = 2x^3 - 3x^2 - 36x + 7$ is increasing or decreasing.

$f'(x) = 6x^2 - 6x - 36 = 6(x - 3)(x + 2)$. Zero at $x = -2, 3$. For $x < -2$: $f' > 0$; $-2 < x < 3$: $f' < 0$; $x > 3$: $f' > 0$. **Increasing on $(-\infty, -2) \cup (3, \infty)$; decreasing on $(-2, 3)$.**

Example 5. Show that $f(x) = e^{2x}$ is increasing on $\mathbb{R}$.

$f'(x) = 2e^{2x} > 0$ for all x .

(3) Maxima and minima

Geometric idea: at the top of a hill (maximum) or bottom of a valley (minimum) the tangent is horizontal, so $f'(x) = 0$.

TESTS

First derivative test: at critical point c , if f' changes from $+$ to $- \rightarrow$ local maximum; from $-$ to $+$ $\rightarrow$ local minimum; no change $\rightarrow$ point of inflection.

Second derivative test: $f'(c) = 0$ and $f''(c) < 0 \rightarrow$ local maximum; $f''(c) > 0 \rightarrow$ local minimum; $f''(c) = 0 \rightarrow$ test fails, use first derivative test.

Absolute max/min on $[a, b]$: evaluate f at critical points and at the end points a, b ; the largest is the absolute maximum, the smallest the absolute minimum.

Example 6. Find local maxima and minima of $f(x) = x^3 - 6x^2 + 9x + 15$.

$f' = 3x^2 - 12x + 9 = 3(x - 1)(x - 3) = 0 \Rightarrow x = 1, 3$. $f'' = 6x - 12$. $f''(1) = -6 < 0 \Rightarrow$ max at $x = 1$, $f(1) = 19$. $f''(3) = 6 > 0 \Rightarrow$ min at $x = 3$, $f(3) = 15$.

Example 7. Find the absolute max and min of $f(x) = 2x^3 - 15x^2 + 36x + 1$ on $[1, 5]$.

$f' = 6x^2 - 30x + 36 = 6(x - 2)(x - 3) \Rightarrow x = 2, 3$. $f(1) = 24$, $f(2) = 29$, $f(3) = 28$, $f(5) = 56$. **Abs max 56 at $x = 5$; abs min 24 at $x = 1$.**

Example 8. Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.

$x + y = 15$, $S = x^2 + (15 - x)^2$. $S' = 2x - 2(15 - x) = 4x - 30 = 0 \Rightarrow x = 7.5$. $S'' = 4 > 0 \Rightarrow$ minimum. Numbers **7.5 and 7.5**.

Example 9. An open box with a square base is made from a square tin sheet of side 18 cm by cutting equal squares from each corner and folding up. Find the side of the square to be cut so that the volume is maximum.

Let x = side cut. $V = x(18 - 2x)^2$. $V' = (18 - 2x)^2 - 4x(18 - 2x) = (18 - 2x)(18 - 6x) = 0 \Rightarrow x = 3$ ($x = 9$ gives no box). V'' at $x = 3$ is negative $\Rightarrow$ maximum. **Cut 3 cm squares**; $V = 3 \times 144 = 432 \text{ cm}^3$.

Example 10. Show that of all rectangles with a given perimeter, the square has the maximum area.

Perimeter $2(x + y) = P \Rightarrow y = P/2 - x$. $A = x(P/2 - x)$. $A' = P/2 - 2x = 0 \Rightarrow x = P/4 \Rightarrow y = P/4$. $A'' = -2 < 0 \Rightarrow$ maximum, and $x = y$ means a square.

Exam Q&A - Calculus I

Q1. Is $f(x) = |x|$ differentiable at $x = 0$? [2]

No. LHD = -1 , RHD = $+1$; they are unequal (though f is continuous at 0).

Q2. Differentiate $e^{\sin x}$. [1]

$e^{\sin x} \cos x$.

Q3. Differentiate $\log(\log x)$. [1]

$1/(x \log x)$.

Q4. Find the rate of change of the area of a circle w.r.t. its radius at $r = 5 \text{ cm}$. [2]

$dA/dr = 2\pi r = 10\pi \text{ cm}^2/\text{cm}$.

Q5. Find the maximum value of $\sin x + \cos x$. [2]

$f' = \cos x - \sin x = 0 \Rightarrow x = \pi/4$; $f(\pi/4) = \sqrt{2}$; $f'' < 0$ there. Maximum $\sqrt{2}$.

XII-7. Integrals

Idea: Integration is the **reverse of differentiation**. If $d/dx F(x) = f(x)$, then $\int f(x) dx = F(x) + C$, where C is any constant (constant of integration). Integration is also used to add up infinitely many tiny pieces — to find areas, volumes, total distance.

STANDARD INTEGRALS

$$\int x^n dx = x^{n+1}/(n+1) + C \quad (n \neq -1) \quad \int 1/x dx = \log|x| + C$$

$$\int e^x dx = e^x + C \quad \int a^x dx = a^x/\log a + C$$

$$\int \sin x dx = -\cos x + C \quad \int \cos x dx = \sin x + C \quad \int \sec^2 x dx = \tan x + C \quad \int \operatorname{cosec}^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C \quad \int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$$

$$\int \tan x dx = \log|\sec x| + C \quad \int \cot x dx = \log|\sin x| + C$$

$$\int \sec x dx = \log|\sec x + \tan x| + C \quad \int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C$$

$$\int 1/\sqrt{1-x^2} dx = \sin^{-1}x + C \quad \int 1/(1+x^2) dx = \tan^{-1}x + C$$

SPECIAL INTEGRALS (SYLLABUS LIST)

$$\int dx/(x^2 - a^2) = (1/2a) \log|(x-a)/(x+a)| + C$$

$$\int dx/(a^2 - x^2) = (1/2a) \log|(a+x)/(a-x)| + C$$

$$\int dx/(x^2 + a^2) = (1/a) \tan^{-1}(x/a) + C$$

$$\int dx/\sqrt{x^2 - a^2} = \log|x + \sqrt{x^2 - a^2}| + C$$

$$\int dx/\sqrt{x^2 + a^2} = \log|x + \sqrt{x^2 + a^2}| + C$$

$$\int dx/\sqrt{a^2 - x^2} = \sin^{-1}(x/a) + C$$

$$\int \sqrt{a^2 - x^2} dx = (x/2)\sqrt{a^2 - x^2} + (a^2/2) \sin^{-1}(x/a) + C$$

$$\int \sqrt{x^2 + a^2} dx = (x/2)\sqrt{x^2 + a^2} + (a^2/2) \log|x + \sqrt{x^2 + a^2}| + C$$

$$\int \sqrt{x^2 - a^2} dx = (x/2)\sqrt{x^2 - a^2} - (a^2/2) \log|x + \sqrt{x^2 - a^2}| + C$$

FOR $AX^2 + BX + C$ IN THE DENOMINATOR

Complete the square: $ax^2 + bx + c = a[(x + b/2a)^2 + (c/a - b^2/4a^2)]$, then use the special integrals above. For $(px + q)/(ax^2 + bx + c)$, write $px + q = A \cdot d/dx(ax^2 + bx + c) + B$, find A and B , and split.

Methods of integration

(1) Substitution: when the integrand contains a function and its derivative, put the function = t .

$$\int 2x \sin(x^2 + 1) dx$$

Put $x^2 + 1 = t$, $2x dx = dt$: $\int \sin t dt = -\cos t + C = -\cos(x^2 + 1) + C$.

$$\int (\log x)^2/x dx$$

$\log x = t$, $dx/x = dt$: $\int t^2 dt = t^3/3 = (\log x)^3/3 + C$.

$$\int e^{\tan^{-1}x} / (1 + x^2) dx$$

$$\tan^{-1}x = t: \int e^t dt = e^{\tan^{-1}x} + C.$$

Trigonometric identities are often needed: $\sin^2 x = (1 - \cos 2x)/2$, $\cos^2 x = (1 + \cos 2x)/2$, $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$.

$$\int \sin^2 x dx$$

$$\int (1 - \cos 2x)/2 dx = x/2 - (\sin 2x)/4 + C.$$

$$\int \sin 3x \cos 4x dx$$

$$\frac{1}{2} \int [\sin 7x + \sin(-x)] dx = \frac{1}{2} [-\cos 7x/7 + \cos x] + C.$$

(2) Partial fractions: for rational functions $P(x)/Q(x)$ with degree $P < \text{degree } Q$.

Denominator factor	Partial fraction form
$(x - a)(x - b)$	$A/(x - a) + B/(x - b)$
$(x - a)^2(x - b)$	$A/(x - a) + B/(x - a)^2 + C/(x - b)$
$(x - a)(x^2 + bx + c)$	$A/(x - a) + (Bx + C)/(x^2 + bx + c)$

$$\int dx / [(x + 1)(x + 2)]$$

$$1/[(x + 1)(x + 2)] = 1/(x + 1) - 1/(x + 2). \text{ Integral} = \log|x + 1| - \log|x + 2| = \log|(x + 1)/(x + 2)| + C.$$

$$\int x dx / [(x - 1)(x - 2)(x - 3)]$$

$$x/[(x-1)(x-2)(x-3)] = A/(x-1) + B/(x-2) + C/(x-3). \text{ Put } x = 1: 1 = A(-1)(-2) \Rightarrow A = 1/2. x = 2: 2 = B(1)(-1) \Rightarrow B = -2. x = 3: 3 = C(2)(1) \Rightarrow C = 3/2. \text{ Integral} = \frac{1}{2} \log|x-1| - 2 \log|x-2| + (3/2) \log|x-3| + C.$$

(3) Integration by parts: $\int u v dx = u \int v dx - \int [u' \int v dx] dx$. Choose u by **ILATE**: Inverse trig, Logarithmic, Algebraic, Trigonometric, Exponential (whichever comes first is u).

$$\int x \sin x dx$$

$$u = x, v = \sin x: x(-\cos x) - \int 1 \cdot (-\cos x) dx = -x \cos x + \sin x + C.$$

$$\int \log x dx$$

$$\text{Write as } \int (\log x) \cdot 1 dx: x \log x - \int (1/x)x dx = x \log x - x + C.$$

$$\int x^2 e^x dx$$

$$x^2 e^x - \int 2x e^x dx = x^2 e^x - 2(xe^x - e^x) = e^x(x^2 - 2x + 2) + C.$$

USEFUL RESULT

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + C. \text{ Example: } \int e^x (\sin x + \cos x) dx = e^x \sin x + C.$$

Special-integral examples

$$\int dx/(x^2 - 16)$$

$$a = 4: (1/8) \log|(x - 4)/(x + 4)| + C.$$

$$\int dx/(x^2 + 6x + 13)$$

$$x^2 + 6x + 13 = (x + 3)^2 + 4. \text{ Integral} = (1/2) \tan^{-1}((x + 3)/2) + C.$$

$$\int dx/\sqrt{7 - 6x - x^2}$$

$$7 - 6x - x^2 = 16 - (x + 3)^2. \text{ Integral} = \sin^{-1}((x + 3)/4) + C.$$

$$\int (x + 2)/(2x^2 + 6x + 5) dx$$

$$x + 2 = A(4x + 6) + B \Rightarrow A = 1/4, B = 2 - 6/4 = 1/2. \text{ Integral} = (1/4)\log|2x^2 + 6x + 5| + (1/2)\int dx/(2x^2 + 6x + 5). \text{ Second: } 2[(x + 3/2)^2 + 1/4] \Rightarrow (1/2)(1/2) \cdot (1/(1/2))\tan^{-1}(2x + 3) = (1/2)\tan^{-1}(2x + 3). \text{ Final: } (1/4)\log|2x^2 + 6x + 5| + (1/2)\tan^{-1}(2x + 3) + C.$$

$$\int \sqrt{4 - x^2} dx$$

$$a = 2: (x/2)\sqrt{4 - x^2} + 2 \sin^{-1}(x/2) + C.$$

Definite integrals

FUNDAMENTAL THEOREM OF CALCULUS (NO PROOF NEEDED)

If $F'(x) = f(x)$, then $\int_a^b f(x) dx = F(b) - F(a)$. The constant C cancels out.

PROPERTIES OF DEFINITE INTEGRALS

$$P1: \int_a^b f(x)dx = \int_a^b f(t)dt \quad P2: \int_a^b = -\int_b^a \quad P3: \int_a^b = \int_a^c + \int_c^b$$

$$P4: \int_a^b f(x)dx = \int_a^b f(a + b - x)dx \quad P5: \int_0^a f(x)dx = \int_0^a f(a - x)dx$$

$$P6: \int_{-a}^a f(x)dx = 2\int_0^a f(x)dx \text{ if } f \text{ is even; } = 0 \text{ if } f \text{ is odd}$$

$$P7: \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(2a - x) = f(x); = 0 \text{ if } f(2a - x) = -f(x)$$

Evaluate $\int_0^{\pi/2} \cos x dx$.

$$[\sin x]_0^{\pi/2} = 1 - 0 = \mathbf{1}.$$

Evaluate $\int_1^2 (4x^3 - 5x^2 + 6x + 9) dx$.

$$[x^4 - 5x^3/3 + 3x^2 + 9x]_1^2 = (16 - 40/3 + 12 + 18) - (1 - 5/3 + 3 + 9) = (46 - 40/3) - (13 - 5/3) = 33 - 35/3 = \mathbf{64/3}.$$

Evaluate $\int_0^{\pi/2} \sqrt{(\sin x)/[\sqrt{(\sin x)} + \sqrt{(\cos x)}]} dx$.

$$\text{Call it } I. \text{ By P5, } I = \int \sqrt{(\cos x)/[\sqrt{(\cos x)} + \sqrt{(\sin x)}]} dx. \text{ Adding: } 2I = \int_0^{\pi/2} 1 dx = \pi/2 \Rightarrow \mathbf{I = \pi/4}.$$

Evaluate $\int_{-1}^1 \sin^5 x \cos^4 x \, dx$.

$f(-x) = -f(x)$ (odd function) $\Rightarrow$ **0**.

Evaluate $\int_0^{\pi/2} \log(\sin x) \, dx$.

$I = \int \log \sin x = \int \log \cos x$ (P5). $2I = \int_0^{\pi/2} \log(\sin x \cos x) \, dx = \int \log(\sin 2x/2) \, dx = \int_0^{\pi/2} \log \sin 2x \, dx - (\pi/2) \log 2$. Put $2x = t$: $\int_0^{\pi/2} \log \sin 2x \, dx = \frac{1}{2} \int_0^{\pi} \log \sin t \, dt = \frac{1}{2} \cdot 2I = I$ (by P7). So $2I = I - (\pi/2) \log 2 \Rightarrow$ **$I = -(\pi/2) \log 2$** .

XII-8. Applications of Integrals (Area)

AREA FORMULAS

Area between curve $y = f(x)$, x-axis and lines $x = a$, $x = b$: $A = \int_a^b |y| \, dx$

Area between curve $x = g(y)$, y-axis and $y = c$, $y = d$: $A = \int_c^d |x| \, dy$

If the curve is below the x-axis, the integral is negative — take its absolute value

Example 1. Find the area enclosed by the circle $x^2 + y^2 = a^2$.

By symmetry, area = $4 \times$ (area in first quadrant) = $4 \int_0^a \sqrt{a^2 - x^2} \, dx = 4[(x/2)\sqrt{a^2 - x^2} - x^2/2 \sin^{-1}(x/a)]_0^a = 4(a^2/2)(\pi/2) =$ **πa^2** .

Example 2. Find the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$.

$y = (b/a)\sqrt{a^2 - x^2}$. Area = $4(b/a) \int_0^a \sqrt{a^2 - x^2} \, dx = 4(b/a)(\pi a^2/4) =$ **πab** .

Example 3. Find the area of the region bounded by $y^2 = 9x$, $x = 2$, $x = 4$ and the x-axis in the first quadrant.

$y = 3\sqrt{x}$. $A = \int_2^4 3x^{1/2} \, dx = 3 \times (2/3)[x^{3/2}]_2^4 = 2(8 - 2\sqrt{2}) =$ **$16 - 4\sqrt{2}$ sq units**.

Example 4. Find the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.

Latus rectum $x = a$. Area = $2 \int_0^a 2\sqrt{ax} \, dx = 4\sqrt{a} (2/3)a^{3/2} =$ **$8a^2/3$** .

Example 5. Find the area under $y = x^2$ from $x = 1$ to $x = 2$ and the x-axis.

$[x^3/3]_1^2 = 8/3 - 1/3 =$ **$7/3$** .

Example 6. Find the area bounded by the line $y = 3x + 2$, the x-axis and $x = -1$, $x = 1$.

Line cuts the x-axis at $x = -2/3$. From -1 to $-2/3$ the line is below the axis. $A = |\int_{-1}^{-2/3} (3x + 2) \, dx| + \int_{-2/3}^1 (3x + 2) \, dx = |[3x^2/2 + 2x]_{-1}^{-2/3}| + [3x^2/2 + 2x]_{-2/3}^1 = |(2/3 - 4/3) - (3/2 - 2)| + (7/2 - (-2/3)) = |-2/3 + 1/2| + 25/6 = 1/6 + 25/6 =$ **$13/3$ sq units**.

XII-9. Differential Equations

DEFINITIONS

- **Differential equation:** an equation involving derivatives, e.g. $dy/dx + y = x$.
- **Order:** the highest derivative present. **Degree:** the power of the highest derivative after the equation is made free of radicals/fractions in derivatives (defined only if it is a polynomial in derivatives).
- **General solution:** contains arbitrary constants (as many as the order). **Particular solution:** obtained by giving the constants specific values using given conditions.

Find the order and degree of $(d^2y/dx^2)^3 + (dy/dx)^2 + \sin(dy/dx) + 1 = 0$.

Order 2; degree **not defined** ($\sin(dy/dx)$ is not a polynomial in the derivative).

Find order and degree of $y''' + 2y'' + y' = 0$.

Order 3, degree 1.

Method 1: Variables separable

If $dy/dx = f(x)g(y)$, write $dy/g(y) = f(x)dx$ and integrate both sides.

Solve $dy/dx = (1 + y^2)/(1 + x^2)$.

$$dy/(1 + y^2) = dx/(1 + x^2) \Rightarrow \tan^{-1}y = \tan^{-1}x + C.$$

Solve $dy/dx = e^{x+y}$.

$$e^{-y}dy = e^x dx \Rightarrow -e^{-y} = e^x + C \Rightarrow e^x + e^{-y} = C.$$

Find the particular solution of $dy/dx = y \tan x$ given $y = 1$ at $x = 0$.

$$dy/y = \tan x dx \Rightarrow \log y = \log \sec x + \log C \Rightarrow y = C \sec x. \text{ At } x = 0: 1 = C. \mathbf{y = \sec x.}$$

Method 2: Homogeneous equations

$dy/dx = F(y/x)$. Put $y = vx$, so $dy/dx = v + x dv/dx$. Then separate variables in v and x .

Solve $(x - y)dy - (x + y)dx = 0$.

$$dy/dx = (x + y)/(x - y). \text{ Put } y = vx: v + x v' = (1 + v)/(1 - v) \Rightarrow x v' = (1 + v^2)/(1 - v) \Rightarrow (1 - v)dv/(1 + v^2) = dx/x. \text{ Integrate: } \tan^{-1}v - \frac{1}{2}\log(1 + v^2) = \log|x| + C. \text{ Put } v = y/x: \mathbf{\tan^{-1}(y/x) = \frac{1}{2}\log(x^2 + y^2) + C.}$$

Solve $x dy/dx = y + x$ (homogeneous).

$$dy/dx = y/x + 1. y = vx: v + x v' = v + 1 \Rightarrow dv = dx/x \Rightarrow v = \log|x| + C \Rightarrow \mathbf{y = x \log|x| + Cx.}$$

Method 3: Linear differential equations

LINEAR D.E.

Form $dy/dx + Py = Q$ (P, Q functions of x): I.F. = $e^{\int P dx}$; solution: $y \times \text{I.F.} = \int (Q \times \text{I.F.}) dx + C$

Form $dx/dy + Px = Q$ (P, Q functions of y): I.F. = $e^{\int P dy}$; solution: $x \times \text{I.F.} = \int (Q \times \text{I.F.}) dy + C$

Solve $dy/dx + 2y = \sin x$.

$P = 2$, I.F. = e^{2x} . $y e^{2x} = \int e^{2x} \sin x dx = e^{2x}(2 \sin x - \cos x)/5 + C \Rightarrow y = (2 \sin x - \cos x)/5 + Ce^{-2x}$.

Solve $x dy/dx + 2y = x^2$ ($x \neq 0$).

$dy/dx + (2/x)y = x$. I.F. = $e^{2 \log x} = x^2$. $yx^2 = \int x^3 dx = x^4/4 + C \Rightarrow y = x^2/4 + C/x^2$.

Solve $dy/dx + y \cot x = 2x + x^2 \cot x$, given $y(\pi/2) = 0$.

I.F. = $e^{\int \cot x} = \sin x$. $y \sin x = \int (2x \sin x + x^2 \cos x) dx = x^2 \sin x + C$ (since $d/dx(x^2 \sin x) = 2x \sin x + x^2 \cos x$). At $x = \pi/2$, $y = 0$: $0 = \pi^2/4 + C \Rightarrow C = -\pi^2/4$. $y \sin x = x^2 \sin x - \pi^2/4$.

Solve $y dx + (x - y^2)dy = 0$.

$dx/dy + x/y = y$. I.F. = $e^{\int dy/y} = y$. $xy = \int y^2 dy = y^3/3 + C \Rightarrow xy = y^3/3 + C$.

Exam Q&A - Integrals and D.E.

Q1. $\int (1/x + e^x) dx = ?$ [1]

$\log|x| + e^x + C$.

Q2. $\int_0^1 dx/(1 + x^2) = ?$ [1]

$\tan^{-1} 1 - 0 = \pi/4$.

Q3. Write the integrating factor of $dy/dx - y = \cos x$. [1]

e^{-x} .

Q4. How many arbitrary constants are in the general solution of a 3rd-order D.E.? [1]

Three.

Q5. Evaluate $\int \sec^2 x / (\tan^2 x + 4) dx$. [2]

$\tan x = t$: $\int dt/(t^2 + 4) = (1/2)\tan^{-1}(\tan x/2) + C$.

XII-10. Vectors

Idea: A **scalar** has only size (magnitude): mass 5 kg, time 10 s, temperature 30 °C. A **vector** has size and direction: displacement "5 km towards north", velocity, force. A vector is drawn as an arrow.

TYPES OF VECTORS

- **Zero vector $\mathbf{0}$:** magnitude 0, no definite direction.
- **Unit vector $\hat{a}$:** magnitude 1; $\hat{a} = \mathbf{a}/|\mathbf{a}|$. $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along x, y, z axes.
- **Equal vectors:** same magnitude and direction.
- **Negative of $\mathbf{a}$:** same magnitude, opposite direction.
- **Collinear / parallel vectors:** lie along the same or parallel lines; $\mathbf{b} = \lambda \mathbf{a}$.
- **Position vector** of point P(x, y, z): $\mathbf{OP} = x\hat{i} + y\hat{j} + z\hat{k}$. $\mathbf{AB} = (\text{position vector of B}) - (\text{position vector of A})$.

VECTOR FORMULAS

$$\mathbf{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, |\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Direction cosines: $l = a_1/|\mathbf{a}|$, $m = a_2/|\mathbf{a}|$, $n = a_3/|\mathbf{a}|$; $l^2 + m^2 + n^2 = 1$. Direction ratios: a_1, a_2, a_3

Section formula (internal, $m : n$): $\mathbf{r} = (m\mathbf{b} + n\mathbf{a})/(m + n)$; external: $(m\mathbf{b} - n\mathbf{a})/(m - n)$; mid-point: $(\mathbf{a} + \mathbf{b})/2$

Dot (scalar) product: $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta = a_1b_1 + a_2b_2 + a_3b_3$

$\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$; $\hat{i} \cdot \hat{i} = 1$, $\hat{i} \cdot \hat{j} = 0$; projection of $\mathbf{a}$ on $\mathbf{b} = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|$

Cross (vector) product: $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin \theta \hat{n} = |\hat{i} \hat{j} \hat{k}; a_1 \ a_2 \ a_3; b_1 \ b_2 \ b_3|$

$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$; $\mathbf{a} \parallel \mathbf{b} \Leftrightarrow \mathbf{a} \times \mathbf{b} = \mathbf{0}$; $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$

Area of parallelogram (sides $\mathbf{a}, \mathbf{b}$) = $|\mathbf{a} \times \mathbf{b}|$; area of triangle = $\frac{1}{2}|\mathbf{a} \times \mathbf{b}|$; unit vector $\perp$ to both = $(\mathbf{a} \times \mathbf{b})/|\mathbf{a} \times \mathbf{b}|$

Geometrical meaning: the dot product measures how much one vector points along another (used for work done = $\mathbf{F} \cdot \mathbf{d}$). The cross product gives a vector perpendicular to both, whose length equals the area of the parallelogram they form (used for torque).

Example 1. Find the unit vector in the direction of $\mathbf{a} = 2\hat{i} + 3\hat{j} + \hat{k}$.

$$|\mathbf{a}| = \sqrt{14}. \hat{a} = (2\hat{i} + 3\hat{j} + \hat{k})/\sqrt{14}.$$

Example 2. Find the direction cosines of the vector $\hat{i} + 2\hat{j} + 3\hat{k}$.

$$|\mathbf{a}| = \sqrt{14}. \text{DCs: } 1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14}.$$

Example 3. Find the position vector of the point dividing the join of P($\hat{i} + 2\hat{j} - \hat{k}$) and Q($-\hat{i} + \hat{j} + \hat{k}$) internally in ratio 2 : 1.

$$\mathbf{r} = [2(-\hat{i} + \hat{j} + \hat{k}) + 1(\hat{i} + 2\hat{j} - \hat{k})]/3 = (-\hat{i} + 4\hat{j} + \hat{k})/3.$$

Example 4. Find the angle between $\mathbf{a} = \hat{i} + \hat{j} - \hat{k}$ and $\mathbf{b} = \hat{i} - \hat{j} + \hat{k}$.

$$\mathbf{a} \cdot \mathbf{b} = 1 - 1 - 1 = -1; |\mathbf{a}| = |\mathbf{b}| = \sqrt{3}. \cos \theta = -1/3 \Rightarrow \theta = \cos^{-1}(-1/3).$$

Example 5. Find λ so that $2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\hat{i} - 2\hat{j} + 3\hat{k}$ are perpendicular.

$$2 - 2\lambda + 3 = 0 \Rightarrow \lambda = 5/2.$$

Example 6. Find the projection of $\vec{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on $\vec{b} = \hat{i} + 2\hat{j} + \hat{k}$.

$$(2 + 6 + 2)/\sqrt{6} = 10/\sqrt{6}.$$

Example 7. Find $\vec{a} \times \vec{b}$ for $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$, and the area of the parallelogram.

$$\vec{a} \times \vec{b} = \hat{i}(1 \cdot (-2) - 3 \cdot 5) - \hat{j}(2 \cdot (-2) - 3 \cdot 3) + \hat{k}(2 \cdot 5 - 1 \cdot 3) = -17\hat{i} + 13\hat{j} + 7\hat{k}. \text{ Area} = \sqrt{(289 + 169 + 49)} = \sqrt{507} \text{ sq units.}$$

Example 8. Find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5), C(1, 5, 5).

$$\vec{AB} = \hat{i} + 2\hat{j} + 3\hat{k}; \vec{AC} = 4\hat{j} + 3\hat{k}. \vec{AB} \times \vec{AC} = \hat{i}(6 - 12) - \hat{j}(3 - 0) + \hat{k}(4 - 0) = -6\hat{i} - 3\hat{j} + 4\hat{k}; \text{ magnitude } \sqrt{61}. \text{ Area} = \frac{1}{2}\sqrt{61} \text{ sq units.}$$

Example 9. If $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\vec{a} \cdot \vec{b} = 4$, find $|\vec{a} - \vec{b}|$.

$$|\vec{a} - \vec{b}|^2 = 4 + 9 - 8 = 5 \Rightarrow \sqrt{5}.$$

XII-11. Three-Dimensional Geometry

LINES IN SPACE

DRs of line through (x_1, y_1, z_1) and (x_2, y_2, z_2) : $x_2 - x_1, y_2 - y_1, z_2 - z_1$; DCs = DRs $\div$ distance

Line through point $\vec{a}$ parallel to $\vec{b}$: **vector form** $\vec{r} = \vec{a} + \lambda\vec{b}$

Cartesian form: $(x - x_1)/a = (y - y_1)/b = (z - z_1)/c$

Line through two points: $\vec{r} = \vec{a} + \lambda(\vec{b} - \vec{a})$; $(x - x_1)/(x_2 - x_1) = (y - y_1)/(y_2 - y_1) = (z - z_1)/(z_2 - z_1)$

Angle between lines with DRs $(a_1, b_1, c_1), (a_2, b_2, c_2)$: $\cos \theta = |a_1a_2 + b_1b_2 + c_1c_2| / (\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2})$

Perpendicular: $a_1a_2 + b_1b_2 + c_1c_2 = 0$; parallel: $a_1/a_2 = b_1/b_2 = c_1/c_2$

Shortest distance between skew lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1, \vec{r} = \vec{a}_2 + \mu\vec{b}_2$: $d = |(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)| / |\vec{b}_1 \times \vec{b}_2|$

Between parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}, \vec{r} = \vec{a}_2 + \mu\vec{b}$: $d = |\vec{b} \times (\vec{a}_2 - \vec{a}_1)| / |\vec{b}|$

Skew lines: lines in space that are neither parallel nor intersecting (like a road and a flyover crossing it). If the shortest distance is 0, the lines intersect.

Example 1. Find the DCs of the line joining $(-2, 4, -5)$ and $(1, 2, 3)$.

$$\text{DRs: } 3, -2, 8; \text{ length } \sqrt{77}. \text{ DCs: } 3/\sqrt{77}, -2/\sqrt{77}, 8/\sqrt{77}.$$

Example 2. Find the vector and Cartesian equations of the line through $(5, 2, -4)$ parallel to $3\hat{i} + 2\hat{j} - 8\hat{k}$.

$$\vec{r} = 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k}). \text{ Cartesian: } (x - 5)/3 = (y - 2)/2 = (z + 4)/(-8).$$

Example 3. Find the angle between $(x - 2)/2 = (y - 1)/5 = (z + 3)/(-3)$ and $(x + 2)/(-1) = (y - 4)/8 = (z - 5)/4$.

$$\cos \theta = |2(-1) + 5(8) + (-3)(4)|/(\sqrt{38} \sqrt{81}) = 26/(9\sqrt{38}) \Rightarrow \theta = \cos^{-1}(26/(9\sqrt{38})).$$

Example 4. Find the shortest distance between $\vec{r} = \hat{i} + \hat{j} + \lambda(2\hat{i} - \hat{j} + \hat{k})$ and $\vec{r} = 2\hat{i} + \hat{j} - \hat{k} + \mu(3\hat{i} - 5\hat{j} + 2\hat{k})$.

$$\vec{b}_1 \times \vec{b}_2 = \hat{i}(-2 + 5) - \hat{j}(4 - 3) + \hat{k}(-10 + 3) = 3\hat{i} - \hat{j} - 7\hat{k}; \text{ magnitude } \sqrt{59}. \vec{a}_2 - \vec{a}_1 = \hat{i} - \hat{k}. \text{ Dot} = 3 + 0 + 7 = 10. d = 10/\sqrt{59}.$$

Example 5. Show that the lines $(x - 5)/7 = (y + 2)/(-5) = z/1$ and $x/1 = y/2 = z/3$ are perpendicular.

$$7(1) + (-5)(2) + 1(3) = 0 \Rightarrow \text{perpendicular.}$$

XII-12. Linear Programming

Idea: A shopkeeper wants maximum profit, a factory wants minimum cost, but both have limits (money, space, time). Linear programming finds the best (optimal) choice under such limits when everything is linear.

TERMS

- **Decision variables:** x, y (quantities to decide).
- **Objective function:** $Z = ax + by$, to be maximised or minimised.
- **Constraints:** linear inequalities (limits), plus non-negativity $x \geq 0, y \geq 0$.
- **Feasible region:** common region satisfying all constraints; every point in it is a **feasible solution**. Points outside are infeasible.
- **Bounded region:** can be enclosed in a circle; **unbounded:** extends infinitely.
- **Optimal feasible solution:** the feasible point giving the best value of Z .

CORNER POINT METHOD

1. Draw each constraint line and shade the feasible region.
2. Find coordinates of all corner points.
3. Evaluate Z at each corner point.
4. Bounded region: largest value = max, smallest = min.
5. Unbounded region: if M is the largest value, check whether $ax + by > M$ has any point in the region; if not, M is the maximum (similarly for minimum with $<$).

Example 1. Maximise $Z = 3x + 4y$ subject to $x + y \leq 4, x \geq 0, y \geq 0$.

Corners: (0,0), (4,0), (0,4). $Z = 0, 12, 16$. **Max $Z = 16$ at (0, 4).**

Example 2. Minimise and maximise $Z = 3x + 9y$ subject to $x + 3y \leq 60$, $x + y \geq 10$, $x \leq y$, $x \geq 0$, $y \geq 0$.

Corner points: A(0, 10), B(5, 5), C(15, 15), D(0, 20). Z: A = 90, B = 60, C = 180, D = 180. **Min $Z = 60$ at (5, 5); Max $Z = 180$ at every point on the segment joining (15, 15) and (0, 20).**

Example 3. A dealer has Rs 5760 to buy fans (Rs 360 each) and sewing machines (Rs 240 each) and space for at most 20 items. Profit is Rs 22 per fan and Rs 18 per machine. How many of each should he buy for maximum profit?

Let x fans, y machines. Max $Z = 22x + 18y$; constraints: $360x + 240y \leq 5760$ (i.e. $3x + 2y \leq 48$), $x + y \leq 20$, $x, y \geq 0$. Intersection of $3x + 2y = 48$ and $x + y = 20$: $x = 8$, $y = 12$. Corners: (0,0), (16,0), (8,12), (0,20). Z: 0, 352, 392, 360. **Buy 8 fans and 12 sewing machines; max profit Rs 392.**

Example 4. Minimise $Z = -3x + 4y$ subject to $x + 2y \leq 8$, $3x + 2y \leq 12$, $x, y \geq 0$.

Intersection: $x + 2y = 8$ and $3x + 2y = 12 \Rightarrow x = 2$, $y = 3$. Corners (0,0), (4,0), (2,3), (0,4). Z: 0, -12, 6, 16. **Min $Z = -12$ at (4, 0).**

XII-13. Probability

KEY FORMULAS

Conditional probability: $P(E|F) = P(E \cap F)/P(F)$, $P(F) \neq 0$

Multiplication theorem: $P(E \cap F) = P(E) P(F|E) = P(F) P(E|F)$

Independent events: $P(E \cap F) = P(E) P(F)$ (then $P(E|F) = P(E)$)

Total probability: if $E_1, \dots, E_n$ partition S: $P(A) = \sum P(E_i) P(A|E_i)$

Bayes' theorem: $P(E_i|A) = P(E_i)P(A|E_i) / \sum P(E_j)P(A|E_j)$

Random variable X with values x_i and probabilities p_i ($\sum p_i = 1$): Mean (expectation) $E(X) = \sum x_i p_i$

DO NOT CONFUSE

Mutually exclusive means $P(E \cap F) = 0$ (cannot happen together). Independent means $P(E \cap F) = P(E)P(F)$ (one does not affect the other). Two events with non-zero probability cannot be both.

Example 1. If $P(E) = 0.6$, $P(F) = 0.3$ and $P(E \cap F) = 0.2$, find $P(E|F)$ and $P(F|E)$.

$P(E|F) = 0.2/0.3 = 2/3$; $P(F|E) = 0.2/0.6 = 1/3$.

Example 2. A die is thrown. E = "number is 4", F = "number is even". Find $P(E|F)$.

$P(E \cap F) = 1/6$, $P(F) = 1/2$. $P(E|F) = 1/3$.

Example 3. A bag has 5 white and 3 black balls. Two balls are drawn one after another without replacement. Find the probability both are white.

$P = (5/8)(4/7) = 5/14$.

Example 4. A and B are independent with $P(A) = 0.3$, $P(B) = 0.4$. Find $P(A \cap B)$, $P(A \cup B)$, $P(\text{neither})$.

$P(A \cap B) = 0.12$. $P(A \cup B) = 0.3 + 0.4 - 0.12 = 0.58$. $P(\text{neither}) = 1 - 0.58 = \mathbf{0.42}$ ($= 0.7 \times 0.6$).

Example 5. Bag I has 3 red, 4 black; Bag II has 5 red, 6 black. A bag is chosen at random and a ball drawn is red. Find the probability it came from Bag II.

$P(E_1) = P(E_2) = 1/2$; $P(R|E_1) = 3/7$; $P(R|E_2) = 5/11$. $P(E_2|R) = (\frac{1}{2} \cdot 5/11)/(\frac{1}{2} \cdot 3/7 + \frac{1}{2} \cdot 5/11) = (5/11)/(3/7 + 5/11) = (5/11)/(68/77) = 35/68$.

Example 6. In a factory, machines A, B, C produce 25%, 35%, 40% of bolts, with 5%, 4%, 2% defective. A bolt chosen at random is defective. Find the probability that it was made by B.

$P(D) = 0.25(0.05) + 0.35(0.04) + 0.40(0.02) = 0.0125 + 0.014 + 0.008 = 0.0345$. $P(B|D) = 0.014/0.0345 = \mathbf{28/69}$.

Example 7. Two coins are tossed. Let X = number of heads. Find the probability distribution and mean.

$X = 0, 1, 2$ with $P = 1/4, 1/2, 1/4$. Mean $= 0(1/4) + 1(1/2) + 2(1/4) = \mathbf{1}$.

Example 8. Find the mean of the number obtained on throwing a die.

$E(X) = (1 + 2 + \dots + 6)/6 = 21/6 = \mathbf{3.5}$.

Example 9. Find k if $P(X = x) = kx$ for $x = 1, 2, 3, 4$ is a probability distribution.

$k(1 + 2 + 3 + 4) = 1 \Rightarrow \mathbf{k = 1/10}$.

Exam Q&A - Vectors, 3-D, LPP, Probability

Q1. Write the value of $\hat{i} \cdot (\hat{j} \times \hat{k})$. [1]

$\hat{i} \cdot \hat{i} = 1$.

Q2. If $\vec{a}$ and $\vec{b}$ are unit vectors at 60° , find $\vec{a} \cdot \vec{b}$. [1]

$\cos 60^\circ = 1/2$.

Q3. Can $1/2, 1/2, 1/2$ be direction cosines of a line? [1]

No, since $(1/4)(3) = 3/4 \neq 1$.

Q4. What is a feasible region in LPP? [1]

The common region determined by all the constraints including non-negativity.

Q5. If A and B are independent and $P(A) = 1/2$, $P(B) = 1/3$, find $P(A|B)$. [1]

$P(A) = 1/2$.

CHAPTER-WISE QUESTION BANK

Mathematics · +2 2nd Year · MCQs, short and long questions with full solutions

HOW TO USE

Attempt each question first; then check. Calculus carries 35 marks, so give it the most practice. Show every step — CHSE awards step marks.

QB-1. Relations and Functions

Key points

- Number of relations on a set with n elements $= 2^{n^2}$; reflexive relations $= 2^{n(n-1)}$.
- Number of functions from A (m elements) to B (n elements) $= n^m$; one-one functions ($m \leq n$) $= {}^n P_m$; bijections ($m = n$) $= n!$.
- Linear functions $ax + b$ ($a \neq 0$) from $\mathbb{R}$ to $\mathbb{R}$ are bijective; x^2 , $|x|$ on $\mathbb{R}$ are neither one-one nor onto.
- Equivalence classes are disjoint and their union is the whole set.

A. Multiple choice questions

1. Let $R = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$ on $\{1, 2, 3\}$. R is (a) reflexive only (b) symmetric (c) equivalence (d) not reflexive

(a) reflexive and transitive but not symmetric; of the options, "reflexive only" (i.e. reflexive but not symmetric) fits.

2. The relation "is parallel to" on the set of lines in a plane is (a) only reflexive (b) only symmetric (c) equivalence (d) none

(c) (taking a line as parallel to itself).

3. $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$ is (a) one-one, not onto (b) onto, not one-one (c) bijective (d) neither

(c).

4. $f : \mathbb{N} \rightarrow \mathbb{N}$, $f(x) = 2x$ is (a) bijective (b) one-one, not onto (c) onto, not one-one (d) neither

(b) odd numbers have no pre-image.

5. Number of equivalence relations on $\{1, 2, 3\}$ containing $(1, 2)$ and $(2, 1)$ is (a) 1 (b) 2 (c) 3 (d) 4

(b) $\{1, 2\}\{3\}$ and $\{1, 2, 3\}$.

6. Number of bijective functions from $\{1, 2, 3\}$ to itself is (a) 3 (b) 6 (c) 9 (d) 27

(b) $3!$.

7. $f(x) = |x|$ from $\mathbb{R}$ to $\mathbb{R}$ is (a) one-one (b) onto (c) bijective (d) neither

(d).

8. If R is symmetric and $(2, 5) \in R$, then (a) $(5, 2) \in R$ (b) $(2, 2) \in R$ (c) $(5, 5) \in R$ (d) none

(a).

B. Short answer questions

1. Show that the relation R on $\mathbb{R}$ defined by $R = \{(a, b) : a \leq b\}$ is reflexive and transitive but not symmetric.

$a \leq a \Rightarrow$ reflexive. $a \leq b$ and $b \leq c \Rightarrow a \leq c \Rightarrow$ transitive. $(1, 2) \in R$ but $(2, 1) \notin R \Rightarrow$ not symmetric.

2. Show that $R = \{(a, b) : |a - b| \text{ is even}\}$ on $A = \{1, 2, 3, 4, 5\}$ is an equivalence relation, and find $[1]$ and $[2]$.

$|a - a| = 0$ even (reflexive); $|a - b| = |b - a|$ (symmetric); if $a - b$ and $b - c$ are even, $a - c = (a - b) + (b - c)$ is even (transitive). $[1] = \{1, 3, 5\}$; $[2] = \{2, 4\}$. Elements of the two classes are not related to each other.

3. Check whether $f : \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 1 + x^2$ is one-one or onto.

$f(1) = f(-1) = 2 \Rightarrow$ not one-one. $f(x) \geq 1$, so 0 has no pre-image $\Rightarrow$ not onto.

4. Show that $f : \mathbb{R} - \{0\} \rightarrow \mathbb{R} - \{0\}$, $f(x) = 1/x$ is bijective.

$1/x_1 = 1/x_2 \Rightarrow x_1 = x_2$ (one-one). For $y \neq 0$, $x = 1/y$ gives $f(x) = y$ (onto).

5. Show that $f : \mathbb{N} \rightarrow \mathbb{N}$, $f(n) = (n + 1)/2$ if n is odd, $n/2$ if n is even, is onto but not one-one.

$f(1) = 1$ and $f(2) = 1 \Rightarrow$ not one-one. For any $m \in \mathbb{N}$, $f(2m) = m \Rightarrow$ onto.

C. Long answer question

Let T be the set of all triangles in a plane with $R = \{(T_1, T_2) : T_1 \text{ is similar to } T_2\}$.

Show that R is an equivalence relation. Consider right triangles T_1 (3, 4, 5), T_2 (5, 12, 13), T_3 (6, 8, 10). Which are related?

Every triangle is similar to itself (reflexive). If $T_1 \sim T_2$ then $T_2 \sim T_1$ (symmetric). If $T_1 \sim T_2$ and $T_2 \sim T_3$, then $T_1 \sim T_3$ (transitive). So R is an equivalence relation. Sides 3, 4, 5 and 6, 8, 10 are proportional (ratio 1 : 2), so T_1 and T_3 are related. T_2 is not similar to either.

QB-2. Inverse Trigonometric Functions

Key points

- $\sin^{-1}(\sin x) = x$ only for $x \in [-\pi/2, \pi/2]$; $\cos^{-1}(\cos x) = x$ only for $x \in [0, \pi]$; $\tan^{-1}(\tan x) = x$ for $x \in (-\pi/2, \pi/2)$.
- $\sin(\sin^{-1}x) = x$ for $x \in [-1, 1]$.
- $2\tan^{-1}x = \sin^{-1}(2x/(1+x^2)) = \cos^{-1}((1-x^2)/(1+x^2)) = \tan^{-1}(2x/(1-x^2))$ (in suitable ranges).

A. Multiple choice questions

1. Principal value of $\sin^{-1}(-1/2)$ is (a) $-\pi/6$ (b) $7\pi/6$ (c) $5\pi/6$ (d) $\pi/6$

(a).

2. Principal value of $\cos^{-1}(-\sqrt{3}/2)$ is (a) $\pi/6$ (b) $5\pi/6$ (c) $-\pi/6$ (d) $7\pi/6$

(b).

3. $\tan^{-1}\sqrt{3} - \sec^{-1}(-2) =$ (a) π (b) $-\pi/3$ (c) $\pi/3$ (d) $2\pi/3$

(b) $\pi/3 - 2\pi/3$.

4. Domain of $\sin^{-1}(2x)$ is (a) $[-1, 1]$ (b) $[-1/2, 1/2]$ (c) $[0, 1]$ (d) $\mathbb{R}$

(b).

5. $\sin(\pi/3 - \sin^{-1}(-1/2)) =$ (a) $1/2$ (b) $1/3$ (c) $1/4$ (d) 1

(d) $\sin(\pi/3 + \pi/6) = \sin(\pi/2)$.

6. $\cos^{-1}(\cos 7\pi/6) =$ (a) $7\pi/6$ (b) $5\pi/6$ (c) $\pi/3$ (d) $\pi/6$

(b).

B. Short answer questions

1. Find the value of $\tan^{-1}(1) + \cos^{-1}(-1/2) + \sin^{-1}(-1/2)$.

$\pi/4 + 2\pi/3 - \pi/6 = (3\pi + 8\pi - 2\pi)/12 = 3\pi/4$.

2. Find the value of $\tan(2 \tan^{-1}(1/5))$.

$\tan 2\theta = 2(1/5)/(1 - 1/25) = (2/5)(25/24) = 5/12$.

3. Write $\tan^{-1}[\sqrt{(1+x^2)} - 1]/x$ ($x \neq 0$) in the simplest form.

Put $x = \tan \theta$: $(\sec \theta - 1)/\tan \theta = (1 - \cos \theta)/\sin \theta = \tan(\theta/2)$. Answer: $\frac{1}{2} \tan^{-1} x$.

4. Find $\sin^{-1}(\sin 2\pi/3)$.

$2\pi/3$ is outside $[-\pi/2, \pi/2]$; $\sin(2\pi/3) = \sin(\pi/3)$. Answer $\pi/3$.

5. Draw the graph of $y = \sin^{-1}x$ and state its domain and range.

An increasing curve from $(-1, -\pi/2)$ through $(0, 0)$ to $(1, \pi/2)$; it is the reflection of $y = \sin x$ (on $[-\pi/2, \pi/2]$) in the line $y = x$. Domain $[-1, 1]$, range $[-\pi/2, \pi/2]$.

QB-3. Matrices

Key points

- Order of $AB = (\text{rows of } A) \times (\text{columns of } B)$. AB defined does not mean BA is defined.
- $(A + B)^2 = A^2 + AB + BA + B^2$ (not $A^2 + 2AB + B^2$ unless $AB = BA$).
- Diagonal elements of a skew-symmetric matrix are zero.
- For any square A : $A + A'$ is symmetric, $A - A'$ is skew-symmetric.
- Number of matrices of order $m \times n$ with entries 0 or 1 $= 2^{mn}$.

A. Multiple choice questions

1. If A is 3×4 and B is 4×2 , the order of AB is (a) 3×2 (b) 4×4 (c) 2×3 (d) not defined

(a).

2. Number of possible orders of a matrix with 12 elements: (a) 4 (b) 6 (c) 3 (d) 12

(b) 1×12 , 2×6 , 3×4 , 4×3 , 6×2 , 12×1 .

3. If $A = [a_{ij}]$ is skew-symmetric, then $a_{22} =$ (a) 1 (b) 0 (c) -1 (d) any

(b).

4. If $A^2 = A$, then $(I + A)^2 - 3A =$ (a) I (b) A (c) O (d) $2A$

(a) $I + 2A + A^2 - 3A = I + 3A - 3A$.

5. The matrix $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ is a (a) identity matrix (b) scalar matrix (c) zero matrix (d) row matrix

(b).

6. $(AB)' =$ (a) $A'B'$ (b) $B'A'$ (c) AB (d) BA

(b).

B. Short answer questions

1. Find x, y, z, w if $\begin{bmatrix} x+y & 2x+z \\ x-y & 2z+w \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ 0 & 10 \end{bmatrix}$.

$x + y = 4$, $x - y = 0 \Rightarrow x = y = 2$. $2x + z = 7 \Rightarrow z = 3$. $2z + w = 10 \Rightarrow w = 4$.

2. If $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$, show that $A + A'$ is symmetric.

$A' = \begin{bmatrix} 2 & 4 \\ 3 & 5 \end{bmatrix}$; $A + A' = \begin{bmatrix} 4 & 7 \\ 7 & 10 \end{bmatrix}$, whose transpose is itself. Symmetric.

3. If $A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$, find α such that $A + A' = I$.

$A + A' = \begin{bmatrix} 2\cos \alpha & 0 \\ 0 & 2\cos \alpha \end{bmatrix} = I \Rightarrow \cos \alpha = 1/2 \Rightarrow \alpha = \pi/3$.

4. For $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, compute AB and BA .

$AB = \begin{bmatrix} 0 \cdot 1 + 1 \cdot 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$; $BA = \begin{bmatrix} 1 \cdot 0 + 0 \cdot 0 & 1 \cdot 1 + 0 \cdot 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. So $AB = O$ but $BA \neq O$, and $AB \neq BA$.

5. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, verify $(A')' = A$ and find $A^2 - 5A$.

$A^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$; $5A = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$; $A^2 - 5A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I$.

C. Long answer questions

1. If $A = \begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 4 \\ -1 & 7 \end{bmatrix}$, find $3A^2 - 2B + I$.

$A^2 = \begin{bmatrix} 4 - 3 & -2 - 2 \\ 6 + 6 & -3 + 4 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 12 & 1 \end{bmatrix}$. $3A^2 = \begin{bmatrix} 3 & -12 \\ 36 & 3 \end{bmatrix}$. $2B = \begin{bmatrix} 0 & 8 \\ -2 & 14 \end{bmatrix}$. $3A^2 - 2B + I = \begin{bmatrix} 3 - 0 + 1 & -12 - 8 \\ 36 + 2 & 3 - 14 + 1 \end{bmatrix} = \begin{bmatrix} 4 & -20 \\ 38 & -10 \end{bmatrix}$.

2. Express $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ as the sum of a symmetric and skew-symmetric matrix.

$A' = A$ (A is already symmetric). So $P = A$, $Q = O$. $A = A + O$.

3. Express $A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 1 \\ 3 & 5 & 7 \end{bmatrix}$ as $P + Q$ (P symmetric, Q skew-symmetric).

$A' = \begin{bmatrix} 1 & 6 & 3 \\ 2 & 8 & 5 \\ 4 & 1 & 7 \end{bmatrix}$. $P = \frac{1}{2}(A + A') = \begin{bmatrix} 1 & 4 & 3.5 \\ 4 & 8 & 3 \\ 3.5 & 3 & 7 \end{bmatrix}$. $Q = \frac{1}{2}(A - A') = \begin{bmatrix} 0 & -2 & 0.5 \\ 2 & 0 & -2 \\ -0.5 & 2 & 0 \end{bmatrix}$. Check $P + Q = A$ ✓.

QB-4. Determinants

Key points

- Interchanging two rows changes the sign of $|A|$; two identical rows $\Rightarrow |A| = 0$.
- Multiplying a row by k multiplies $|A|$ by k .
- $R_i \rightarrow R_i + kR_j$ does not change $|A|$.
- A singular matrix has $|A| = 0$ and no inverse.
- $|A^{-1}| = 1/|A|$; $\text{adj}(AB) = \text{adj } B \cdot \text{adj } A$.

A. Multiple choice questions

1. $\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix} =$ (a) 18 (b) -18 (c) 22 (d) 2

(a) $-2 + 20$.

2. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then $x =$ (a) 6 (b) ± 6 (c) -6 (d) 0

(b) $x^2 - 36 = 36 - 36 \Rightarrow x^2 = 36$.

3. If A is 3×3 and $|A| = 5$, $|\text{adj } A| =$ (a) 5 (b) 25 (c) 125 (d) 15

(b) 5^2 .

4. A system $AX = B$ has a unique solution if (a) $|A| = 0$ (b) $|A| \neq 0$ (c) $B = O$ (d) $A = I$ (b).

5. Value of $|1 \ a \ b + c; 1 \ b \ c + a; 1 \ c \ a + b|$ is (a) 0 (b) 1 (c) $a + b + c$ (d) abc

(a) $C_3 \rightarrow C_3 + C_2$ gives a column of $(a + b + c)$, proportional to C_1 .

B. Short answer questions

1. Find the minors and cofactors of the elements of the first row of $|1 \ 0 \ 4; 3 \ 5 \ -1; 0 \ 1 \ 2|$.

$M_{11} = 10 + 1 = 11$, $A_{11} = 11$. $M_{12} = 6 - 0 = 6$, $A_{12} = -6$. $M_{13} = 3 - 0 = 3$, $A_{13} = 3$.
 $|A| = 1(11) + 0 + 4(3) = 23$.

2. Find the area of the triangle with vertices $(-2, -3)$, $(3, 2)$, $(-1, -8)$.

$\frac{1}{2}|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| = \frac{1}{2}|-2(10) + 3(-5) + (-1)(-5)| = \frac{1}{2}|-20 - 15 + 5| = 15 \text{ sq units}$.

3. Using determinants, find the equation of the line joining $(1, 2)$ and $(3, 6)$.

$|x \ y \ 1; 1 \ 2 \ 1; 3 \ 6 \ 1| = 0 \Rightarrow x(2 - 6) - y(1 - 3) + (6 - 6) = 0 \Rightarrow -4x + 2y = 0 \Rightarrow y = 2x$.

4. Verify $A(\text{adj } A) = |A|I$ for $A = [2 \ 3; -4 \ -6]$.

$|A| = -12 + 12 = 0$. $\text{adj } A = [-6 \ -3; 4 \ 2]$. $A(\text{adj } A) = [-12 + 12, -6 + 6; 24 - 24, 12 - 12] = O = 0 \cdot I$ ✓.

5. Check consistency: $2x - y = 5$, $x + y = 4$.

$|A| = 2 + 1 = 3 \neq 0 \Rightarrow$ consistent with a unique solution ($x = 3$, $y = 1$).

C. Long answer questions

1. Prove $|1 \ 1 \ 1; a \ b \ c; a^3 \ b^3 \ c^3| = (a - b)(b - c)(c - a)(a + b + c)$.

$C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 - C_1$: $|1 \ 0 \ 0; a \ b - a \ c - a; a^3 \ b^3 - a^3 \ c^3 - a^3|$. Take $(b - a)$ and $(c - a)$ common from C_2 and C_3 : $(b - a)(c - a)|1 \ 0 \ 0; a \ 1 \ 1; a^3 \ b^2 + ab + a^2 \ c^2 + ca + a^2|$. Expand along R_1 : $(b - a)(c - a)[(c^2 + ca + a^2) - (b^2 + ab + a^2)] = (b - a)(c - a)[(c - b)(c + b) + a(c - b)] = (b - a)(c - a)(c - b)(a + b + c)$. Since $(b - a)(c - b) = (a - b)(b - c)$, the value is $(a - b)(b - c)(c - a)(a + b + c)$.

2. Solve by matrix method: $x - y + 2z = 7$, $3x + 4y - 5z = -5$, $2x - y + 3z = 12$.

$A = [1 \ -1 \ 2; 3 \ 4 \ -5; 2 \ -1 \ 3]$. $|A| = 1(12 - 5) + 1(9 + 10) + 2(-3 - 8) = 7 + 19 - 22 = 4$.

Cofactors: $A_{11} = 7$, $A_{12} = -19$, $A_{13} = -11$; $A_{21} = -(-3 + 2) = 1$, $A_{22} = 3 - 4 = -1$, $A_{23} = -(-1 + 2) = -1$; $A_{31} = 5 - 8 = -3$, $A_{32} = -(-5 - 6) = 11$, $A_{33} = 4 + 3 = 7$.

$\text{adj } A = [7 \ 1 \ -3; -19 \ -1 \ 11; -11 \ -1 \ 7]$. $X = (1/4)[49 - 5 - 36; -133 + 5 + 132; -77 + 5 + 84] = (1/4)[8; 4; 12] \Rightarrow \mathbf{x = 2, y = 1, z = 3}$.

3. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs 60; 2 kg onion, 4 kg wheat and 6 kg rice is Rs 90; 6 kg onion, 2 kg wheat and 3 kg rice is Rs 70. Find the cost per kg of each by matrix method.

$4x + 3y + 2z = 60$; $2x + 4y + 6z = 90$; $6x + 2y + 3z = 70$. $|A| = 4(12 - 12) - 3(6 - 36) + 2(4 - 24) = 0 + 90 - 40 = 50$. Solving (cofactors: $A_{11} = 0$, $A_{12} = 30$, $A_{13} = -20$;

$A_{21} = -5$, $A_{22} = 0$, $A_{23} = 10$; $A_{31} = 10$, $A_{32} = -20$, $A_{33} = 10$): $x = (0 \cdot 60 - 5 \cdot 90 + 10 \cdot 70)/50 = 250/50 = 5$; $y = (30 \cdot 60 + 0 - 20 \cdot 70)/50 = 400/50 = 8$; $z = (-20 \cdot 60 + 10 \cdot 90 + 10 \cdot 70)/50 = 400/50 = 8$. **Onion Rs 5, wheat Rs 8, rice Rs 8 per kg.**

QB-5. Continuity and Differentiability

Key points

- Continuity at a : $\lim_{x \rightarrow a^-} f = \lim_{x \rightarrow a^+} f = f(a)$.
- Differentiable $\Rightarrow$ continuous, but not conversely ($|x|$ at 0).
- Chain rule: $d/dx f(g(x)) = f'(g(x)) \cdot g'(x)$.
- Implicit: differentiate y -terms and multiply by dy/dx .
- Use logarithmic differentiation for u^v forms and long products/quotients.
- Useful substitutions: $x = \tan \theta$ for $(1 - x^2)/(1 + x^2)$, $2x/(1 + x^2)$, $2x/(1 - x^2)$; $x = \sin \theta$ or $\cos \theta$ for $\sqrt{1 - x^2}$.

A. Multiple choice questions

- $f(x) = [x]$ is discontinuous at (a) all real x (b) all integers (c) $x = 0$ only (d) nowhere (b).
- $d/dx(\sin^2 x) =$ (a) $2 \sin x$ (b) $\sin 2x$ (c) $\cos^2 x$ (d) $2 \cos x$ (b).
- $d/dx(e^{x^2}) =$ (a) e^{x^2} (b) $2x e^{x^2}$ (c) $x^2 e^x$ (d) $2e^x$ (b).
- $d/dx(\log \sin x) =$ (a) $\cot x$ (b) $\tan x$ (c) $1/\sin x$ (d) $-\cot x$ (a).
- If $y = \tan^{-1} x$, then $(1 + x^2) dy/dx =$ (a) 0 (b) 1 (c) x (d) $2x$ (b).
- If $x = t^2$, $y = t^3$, $dy/dx =$ (a) $3t/2$ (b) $2/(3t)$ (c) $3t^2$ (d) t (a).
- d^2y/dx^2 of $y = e^{3x}$ is (a) $3e^{3x}$ (b) $9e^{3x}$ (c) e^{3x} (d) $6e^{3x}$ (b).
- $d/dx(x^x)$ at $x = 1$ is (a) 0 (b) 1 (c) e (d) 2 (b) $1^1(1 + \log 1) = 1$.
- $f(x) = |x - 2|$ is not differentiable at (a) 0 (b) 2 (c) -2 (d) everywhere differentiable (b).
- $d/dx(\log_{10} x) =$ (a) $1/x$ (b) $1/(x \log 10)$ (c) $\log 10/x$ (d) $10/x$ (b).

B. Short answer questions

1. Find a and b if $f(x) = 5$ ($x \leq 2$), $ax + b$ ($2 < x < 10$), 21 ($x \geq 10$) is continuous.

At $x = 2$: $2a + b = 5$. At $x = 10$: $10a + b = 21$. Subtract: $8a = 16 \Rightarrow a = 2$, $b = 1$.

2. Show that $f(x) = |x|$ is continuous but not differentiable at $x = 0$.

LHL = RHL = $f(0) = 0 \Rightarrow$ continuous. LHD = $\lim(|h| - 0)/h$ as $h \rightarrow 0^- = -1$; RHD = $+1$. Unequal $\Rightarrow$ not differentiable.

3. Differentiate $\cos(\sqrt{x})$.

$$-\sin \sqrt{x} \cdot 1/(2\sqrt{x}).$$

4. Differentiate $\sec(\tan \sqrt{x})$.

$$\sec(\tan \sqrt{x}) \tan(\tan \sqrt{x}) \cdot \sec^2 \sqrt{x} \cdot 1/(2\sqrt{x}).$$

5. Find dy/dx if $xy + y^2 = \tan x + y$.

$$y + xy' + 2yy' = \sec^2 x + y' \Rightarrow y'(x + 2y - 1) = \sec^2 x - y \Rightarrow dy/dx = (\sec^2 x - y)/(x + 2y - 1).$$

6. Differentiate $\cos^{-1}((1 - x^2)/(1 + x^2))$, $0 < x < 1$.

$$x = \tan \theta \Rightarrow \cos^{-1}(\cos 2\theta) = 2\theta = 2 \tan^{-1} x \Rightarrow 2/(1 + x^2).$$

7. Differentiate $e^x \sin x + \log(\cos x)$.

$$e^x(\sin x + \cos x) - \tan x.$$

8. Find dy/dx if $x = a \cos \theta$, $y = b \sin \theta$.

$$dy/dx = b \cos \theta / (-a \sin \theta) = -(b/a) \cot \theta.$$

9. If $y = 5 \cos x - 3 \sin x$, show $y'' + y = 0$.

$$y' = -5 \sin x - 3 \cos x; y'' = -5 \cos x + 3 \sin x = -y.$$

10. Differentiate $(\log x)^x$.

$$\log y = x \log(\log x) \Rightarrow y'/y = \log(\log x) + 1/\log x \Rightarrow y' = (\log x)^x [\log(\log x) + 1/\log x].$$

C. Long answer questions**1. If $y = (\tan^{-1} x)^2$, show that $(1 + x^2)^2 y'' + 2x(1 + x^2) y' = 2$.**

$$y' = 2 \tan^{-1} x / (1 + x^2) \Rightarrow (1 + x^2) y' = 2 \tan^{-1} x. \text{ Differentiate: } (1 + x^2) y'' + 2xy' = 2/(1 + x^2). \text{ Multiply by } (1 + x^2): (1 + x^2)^2 y'' + 2x(1 + x^2) y' = 2.$$

2. If $x^y = e^{x-y}$, prove $dy/dx = \log x / (1 + \log x)^2$.

$$y \log x = x - y \Rightarrow y(1 + \log x) = x \Rightarrow y = x/(1 + \log x). \text{ } dy/dx = [(1 + \log x) - x(1/x)]/(1 + \log x)^2 = \log x / (1 + \log x)^2.$$

3. If $y = e^{a \cos^{-1}x}$, $-1 < x < 1$, show $(1 - x^2)y'' - xy' - a^2y = 0$.

$y' = e^{a \cos^{-1}x} \cdot (-a/\sqrt{1-x^2}) \Rightarrow \sqrt{1-x^2} y' = -ay$. Square: $(1-x^2)y'^2 = a^2y^2$.

Differentiate: $(1-x^2)2y'y'' - 2xy'^2 = 2a^2yy'$. Divide by $2y'$: $(1-x^2)y'' - xy' - a^2y = 0$.

4. Find dy/dx if $y = x^{\sin x} + (\sin x)^{\cos x}$.

$u = x^{\sin x}$: $u' = x^{\sin x}(\cos x \log x + \sin x/x)$. $v = (\sin x)^{\cos x}$: $\log v = \cos x \log \sin x \Rightarrow v' = (\sin x)^{\cos x}(-\sin x \log \sin x + \cos x \cot x)$. $dy/dx = u' + v'$.

QB-6. Applications of Derivatives

Key points

- Rate of change: $dy/dt = (dy/dx)(dx/dt)$. Marginal cost/revenue = derivative of cost/revenue.
- Increasing where $f' > 0$; decreasing where $f' < 0$.
- Local max: $f'(c) = 0$, $f''(c) < 0$; local min: $f''(c) > 0$.
- Absolute extrema on $[a, b]$: compare values at critical points and end points.
- Useful geometry: sphere $V = 4\pi r^3/3$, $S = 4\pi r^2$; cylinder $V = \pi r^2 h$, $CSA = 2\pi r h$; cone $V = \pi r^2 h/3$.

A. Multiple choice questions

1. Rate of change of area of a square of side x w.r.t. x at $x = 4$ is (a) 4 (b) 8 (c) 16 (d) 2
(b) $2x$.
2. $f(x) = 3x + 17$ is (a) increasing on $\mathbb{R}$ (b) decreasing (c) neither (d) constant
(a).
3. $f(x) = x^2$ is decreasing on (a) $(0, \infty)$ (b) $(-\infty, 0)$ (c) $\mathbb{R}$ (d) nowhere
(b).
4. The minimum value of $x^2 - 4x + 7$ is (a) 7 (b) 3 (c) 4 (d) 0
(b) at $x = 2$.
5. Max value of $\sin x \cos x$ is (a) 1 (b) $1/2$ (c) $\sqrt{2}$ (d) 2
(b).
6. $f(x) = x^3$ has at $x = 0$ (a) max (b) min (c) point of inflection (d) none
(c).

B. Short answer questions

1. The volume of a cube increases at $8 \text{ cm}^3/\text{s}$. How fast is the surface area increasing when the edge is 12 cm ?

$V = x^3 \Rightarrow 3x^2 dx/dt = 8 \Rightarrow dx/dt = 8/432 = 1/54$. $S = 6x^2 \Rightarrow dS/dt = 12x dx/dt = 144/54 = 8/3 \text{ cm}^2/\text{s}$.

2. A stone dropped into a pond makes circular waves; the radius increases at 5 cm/s. How fast is the area increasing when $r = 8$ cm?

$$dA/dt = 2\pi r dr/dt = 2\pi(8)(5) = \mathbf{80\pi \text{ cm}^2/s}.$$

3. Total revenue $R(x) = 3x^2 + 36x + 5$. Find the marginal revenue at $x = 5$.

$$R'(x) = 6x + 36 = 66.$$

4. Find the intervals where $f(x) = \sin x + \cos x$, $0 \leq x \leq 2\pi$, is increasing.

$$f' = \cos x - \sin x > 0 \text{ when } \tan x < 1 \text{ in appropriate quadrants: } [0, \pi/4) \text{ and } (5\pi/4, 2\pi].$$

5. Show that $f(x) = \log x$ is increasing on $(0, \infty)$.

$$f' = 1/x > 0 \text{ for } x > 0.$$

6. Find the local maximum and minimum values of $f(x) = x^3 - 3x$.

$$f' = 3x^2 - 3 = 0 \Rightarrow x = \pm 1. f'' = 6x. \text{ At } x = -1: f'' < 0 \Rightarrow \text{local max } f(-1) = 2. \text{ At } x = 1: \text{local min } f(1) = -2.$$

C. Long answer questions

1. Show that the right circular cylinder of given surface area (closed) and maximum volume has height equal to the diameter of the base.

$$S = 2\pi r^2 + 2\pi rh \Rightarrow h = (S - 2\pi r^2)/(2\pi r). V = \pi r^2 h = (Sr - 2\pi r^3)/2. dV/dr = (S - 6\pi r^2)/2 = 0 \Rightarrow S = 6\pi r^2. \text{ Then } h = (6\pi r^2 - 2\pi r^2)/(2\pi r) = 2r. d^2V/dr^2 = -6\pi r < 0 \Rightarrow \text{maximum. So } h = 2r = \text{diameter.}$$

2. A wire of length 28 m is cut into two pieces; one is bent into a square and the other into a circle. Where should it be cut so that the combined area is minimum?

$$\text{Let } x \text{ m make the square (side } x/4) \text{ and } 28 - x \text{ the circle (} r = (28 - x)/2\pi). A = x^2/16 + (28 - x)^2/(4\pi). dA/dx = x/8 - (28 - x)/(2\pi) = 0 \Rightarrow \pi x = 4(28 - x) \Rightarrow x = 112/(\pi + 4). A'' = 1/8 + 1/(2\pi) > 0 \Rightarrow \text{minimum. Square piece} = \mathbf{112/(\pi + 4) \text{ m} \approx 15.68 \text{ m}}; \text{ circle piece} = 28\pi/(\pi + 4) \text{ m} \approx 12.32 \text{ m}.$$

3. Find the point on the curve $x^2 = 2y$ nearest to $(0, 5)$.

$$\text{Point } (x, x^2/2). D^2 = x^2 + (x^2/2 - 5)^2. \text{ Put } t = x^2: D^2 = t + (t/2 - 5)^2; \text{ derivative } 1 + (t/2 - 5) = 0 \Rightarrow t = 8 \Rightarrow x = \pm 2\sqrt{2}, y = 4. \text{ Nearest points } (\pm 2\sqrt{2}, 4).$$

4. Find the absolute max and min of $f(x) = 12x^{4/3} - 6x^{1/3}$ on $[-1, 1]$.

$$f' = 16x^{1/3} - 2x^{-2/3} = (16x - 2)/x^{2/3} \Rightarrow \text{critical points } x = 1/8 \text{ (and } x = 0, \text{ where } f' \text{ is undefined)}. f(-1) = 12 + 6 = 18; f(0) = 0; f(1/8) = 12(1/16) - 6(1/2) = 0.75 - 3 = -9/4; f(1) = 6. \text{ Absolute max } \mathbf{18 \text{ at } x = -1}; \text{ absolute min } \mathbf{-9/4 \text{ at } x = 1/8}.$$

5. A ladder 5 m long leans against a wall; its foot slides away at 2 cm/s. How fast is the height on the wall decreasing when the foot is 4 m from the wall?

$$x^2 + y^2 = 25, x = 4, y = 3. x \, dx/dt + y \, dy/dt = 0 \Rightarrow dy/dt = -(4/3)(2) = -8/3 \text{ cm/s.}$$

QB-7. Integrals (Indefinite and Definite)

Key points

- Integration is the reverse of differentiation: if $d/dx F(x) = f(x)$, then $\int f(x) dx = F(x) + C$, where C is the constant of integration.
- Always write "+ C" in indefinite integrals. In definite integrals the constant cancels, so it is not written.
- **Substitution:** if the integrand contains a function and its derivative, put the function = t . Example: in $\int 2x \cos(x^2) dx$ put $x^2 = t$.
- **Partial fractions:** used for rational functions $P(x)/Q(x)$ with degree $P < \text{degree } Q$. If degree $P \geq \text{degree } Q$, divide first.
- **By parts:** $\int u v dx = u \int v dx - \int (u' \int v dx) dx$. Choose u by the ILATE order: Inverse trig, Logarithmic, Algebraic, Trigonometric, Exponential.
- **Special forms:** $\int dx/(x^2 \pm a^2)$, $\int dx/\sqrt{a^2 - x^2}$, $\int \sqrt{x^2 + a^2} dx$ etc. For a quadratic $ax^2 + bx + c$, first complete the square.
- **Definite integral:** $\int_a^b f(x) dx = F(b) - F(a)$ (Second Fundamental Theorem).
- When you substitute in a definite integral, change the limits too.

STANDARD INTEGRALS TO MEMORISE

$$\int x^n dx = x^{n+1}/(n+1) + C \quad (n \neq -1); \int dx/x = \log|x| + C; \int e^x dx = e^x + C; \int a^x dx = a^x/\log a + C$$

$$\int \sin x dx = -\cos x + C; \int \cos x dx = \sin x + C; \int \sec^2 x dx = \tan x + C; \int \operatorname{cosec}^2 x dx = -\cot x + C$$

$$\int \tan x dx = \log|\sec x| + C; \int \cot x dx = \log|\sin x| + C; \int \sec x dx = \log|\sec x + \tan x| + C; \int \operatorname{cosec} x dx = \log|\operatorname{cosec} x - \cot x| + C$$

$$\int dx/(x^2 + a^2) = (1/a) \tan^{-1}(x/a) + C; \int dx/(x^2 - a^2) = (1/2a) \log|(x-a)/(x+a)| + C; \int dx/(a^2 - x^2) = (1/2a) \log|(a+x)/(a-x)| + C$$

$$\int dx/\sqrt{a^2 - x^2} = \sin^{-1}(x/a) + C; \int dx/\sqrt{x^2 + a^2} = \log|x + \sqrt{x^2 + a^2}| + C; \int dx/\sqrt{x^2 - a^2} = \log|x + \sqrt{x^2 - a^2}| + C$$

$$\int \sqrt{a^2 - x^2} dx = (x/2)\sqrt{a^2 - x^2} + (a^2/2) \sin^{-1}(x/a) + C$$

$$\int \sqrt{x^2 + a^2} dx = (x/2)\sqrt{x^2 + a^2} + (a^2/2) \log|x + \sqrt{x^2 + a^2}| + C$$

$$\int \sqrt{x^2 - a^2} dx = (x/2)\sqrt{x^2 - a^2} - (a^2/2) \log|x + \sqrt{x^2 - a^2}| + C$$

$$\int e^x [f(x) + f'(x)] dx = e^x f(x) + C$$

PROPERTIES OF DEFINITE INTEGRALS

$$P1: \int_a^b f(x) dx = -\int_b^a f(x) dx. \quad P2: \int_a^b f = \int_a^c f + \int_c^b f.$$

$$P3: \int_a^b f(x) dx = \int_a^b f(a+b-x) dx. \quad P4: \int_0^a f(x) dx = \int_0^a f(a-x) dx.$$

$$P5: \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \text{ if } f \text{ is even; } = 0 \text{ if } f \text{ is odd.}$$

$$P6: \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx \text{ if } f(2a-x) = f(x); = 0 \text{ if } f(2a-x) = -f(x).$$

HOW TO CHOOSE THE METHOD

(1) Is it a standard form? Write the answer directly. (2) Is the derivative of some part present? Use substitution. (3) Is it a fraction of two polynomials? Use partial fractions. (4) Is it a product of two different types of functions (like $x \cdot e^x$, $x \cdot \sin x$, $\log x$)? Use by parts. (5) Does it contain a quadratic under a root or in a denominator? Complete the square.

A. Multiple choice questions

1. $\int dx/(1+x^2) =$ (a) $\log(1+x^2) + C$ (b) $\tan^{-1}x + C$ (c) $\sin^{-1}x + C$ (d) $\cot x + C$
(b).

2. $\int e^x(\tan x + \sec^2 x) dx =$ (a) $e^x \sec^2 x + C$ (b) $e^x \tan x + C$ (c) $e^x \sec x + C$ (d) $e^x + C$
(b) It is of the form $e^x[f + f']$ with $f = \tan x$.

3. $\int_0^{\pi/2} \sin x dx =$ (a) 0 (b) 1 (c) -1 (d) $\pi/2$
(b) $[-\cos x]_0^{\pi/2} = 0 + 1 = 1$.

4. $\int_{-1}^1 x^3 dx =$ (a) $1/2$ (b) 0 (c) $1/4$ (d) 2
(b) x^3 is an odd function.

5. $\int \tan x dx =$ (a) $\sec^2 x + C$ (b) $\log|\cos x| + C$ (c) $\log|\sec x| + C$ (d) $-\log|\sec x| + C$
(c).

6. $\int_0^1 x e^x dx =$ (a) e (b) 1 (c) $e - 1$ (d) 0
(b) $[x e^x - e^x]_0^1 = 0 - (-1) = 1$.

7. $\int dx/\sqrt{1-x^2} =$ (a) $\cos^{-1}x + C$ (b) $\sin^{-1}x + C$ (c) $\tan^{-1}x + C$ (d) $\log x + C$
(b).

8. $\int_0^{\pi/2} \sin x/(\sin x + \cos x) dx =$ (a) π (b) $\pi/2$ (c) $\pi/4$ (d) 0
(c) Using P4, $I = \int \cos x/(\cos x + \sin x) dx$; adding, $2I = \pi/2$.

9. $\int 2x/(1+x^2) dx =$ (a) $\tan^{-1}x + C$ (b) $\log(1+x^2) + C$ (c) $2 \log x + C$ (d) $x^2 + C$
(b) The numerator is the derivative of the denominator.

10. $\int_0^{2\pi} \sin^5 x dx =$ (a) 0 (b) 1 (c) π (d) 2π
(a) By P6, since $\sin^5(2\pi - x) = -\sin^5 x$.

B. Short answer questions

1. Evaluate $\int (2x+3)/(x^2+3x+5) dx$.

Put $t = x^2 + 3x + 5$, so $dt = (2x+3) dx$. The integral becomes $\int dt/t = \log|t| + C = \log|x^2 + 3x + 5| + C$.

2. Evaluate $\int \sin^3 x \cos x dx$.

Put $\sin x = t$, $\cos x dx = dt$. $\int t^3 dt = t^4/4 + C = \sin^4 x/4 + C$.

3. Evaluate $\int x \log x \, dx$.

By ILATE, $u = \log x$ (L comes before A), $v = x$.

$$\int x \log x \, dx = \log x \cdot x^2/2 - \int (1/x)(x^2/2) \, dx = (x^2/2) \log x - \int x/2 \, dx = \mathbf{(x^2/2) \log x - x^2/4 + C}.$$

4. Evaluate $\int e^x(\sin x + \cos x) \, dx$.

Take $f(x) = \sin x$; then $f'(x) = \cos x$. So the integral is $e^x f(x) + C = \mathbf{e^x \sin x + C}$.

5. Evaluate $\int dx/(x^2 + 4x + 8)$.

Complete the square: $x^2 + 4x + 8 = (x + 2)^2 + 2^2$. Using $\int dx/(X^2 + a^2) = (1/a) \tan^{-1}(X/a)$: answer $\mathbf{(1/2) \tan^{-1}((x + 2)/2) + C}$.

6. Evaluate $\int dx/[x(x + 1)]$.

$1/[x(x + 1)] = 1/x - 1/(x + 1)$. So the integral $= \log|x| - \log|x + 1| + C = \mathbf{\log|x/(x + 1)| + C}$.

7. Evaluate $\int_0^{\pi/4} \tan^2 x \, dx$.

$$\tan^2 x = \sec^2 x - 1. \int = [\tan x - x]_0^{\pi/4} = (1 - \pi/4) - 0 = \mathbf{1 - \pi/4}.$$

8. Evaluate $\int_0^2 |x - 1| \, dx$.

For $0 \leq x \leq 1$, $|x - 1| = 1 - x$; for $1 \leq x \leq 2$, $|x - 1| = x - 1$.

$$\int_0^1 (1 - x) \, dx = 1/2 \text{ and } \int_1^2 (x - 1) \, dx = 1/2. \text{ Total} = \mathbf{1}.$$

9. Evaluate $\int \sin^2 x \, dx$.

$$\sin^2 x = (1 - \cos 2x)/2. \int = x/2 - (\sin 2x)/4 + C. \text{ Answer } \mathbf{x/2 - (\sin 2x)/4 + C}.$$

10. Evaluate $\int dx/(1 + \cos x)$.

$1 + \cos x = 2\cos^2(x/2)$, so the integrand $= (1/2) \sec^2(x/2)$. $\int = (1/2) \cdot \tan(x/2)/(1/2) + C = \mathbf{\tan(x/2) + C}$.

C. Long answer questions

1. Evaluate $\int (3x - 1)/[(x - 1)(x - 2)(x - 3)] dx$.

Step 1 (set up): $(3x - 1)/[(x - 1)(x - 2)(x - 3)] = A/(x - 1) + B/(x - 2) + C/(x - 3)$.

Step 2 (find constants by putting zeros of factors):

Multiply out: $3x - 1 = A(x - 2)(x - 3) + B(x - 1)(x - 3) + C(x - 1)(x - 2)$.

$x = 1: 2 = A(-1)(-2) = 2A \Rightarrow A = 1$.

$x = 2: 5 = B(1)(-1) = -B \Rightarrow B = -5$.

$x = 3: 8 = C(2)(1) = 2C \Rightarrow C = 4$.

Step 3 (integrate): $f = \log|x - 1| - 5 \log|x - 2| + 4 \log|x - 3| + C$.

Answer: **$\log|x - 1| - 5 \log|x - 2| + 4 \log|x - 3| + C$** .

2. Prove that $\int_0^\pi x \sin x/(1 + \cos^2 x) dx = \pi^2/4$.

Let $I = \int_0^\pi x \sin x/(1 + \cos^2 x) dx$. — (1)

Using P4 (replace x by $\pi - x$): $\sin(\pi - x) = \sin x$ and $\cos(\pi - x) = -\cos x$, so

$I = \int_0^\pi (\pi - x) \sin x/(1 + \cos^2 x) dx$. — (2)

Add (1) and (2): $2I = \pi \int_0^\pi \sin x/(1 + \cos^2 x) dx$.

Put $\cos x = t$, $-\sin x dx = dt$. When $x = 0$, $t = 1$; when $x = \pi$, $t = -1$.

$2I = \pi \int_1^{-1} (-dt)/(1 + t^2) = \pi \int_{-1}^1 dt/(1 + t^2) = \pi [\tan^{-1} t]_{-1}^1 = \pi(\pi/4 + \pi/4) = \pi^2/2$.

Hence **$I = \pi^2/4$** . Proved.

3. Evaluate $\int_0^{\pi/2} \log \sin x dx$.

Let $I = \int_0^{\pi/2} \log \sin x dx$. By P4, $I = \int_0^{\pi/2} \log \cos x dx$.

Add: $2I = \int_0^{\pi/2} \log(\sin x \cos x) dx = \int_0^{\pi/2} \log(\sin 2x/2) dx = \int_0^{\pi/2} \log \sin 2x dx - (\pi/2) \log 2$.

In $J = \int_0^{\pi/2} \log \sin 2x dx$ put $2x = t$, $dx = dt/2$; limits 0 to π : $J = (1/2) \int_0^\pi \log \sin t dt = (1/2) \cdot 2 \int_0^{\pi/2} \log \sin t dt = I$ (by P6, since $\sin(\pi - t) = \sin t$).

So $2I = I - (\pi/2) \log 2 \Rightarrow$ **$I = -(\pi/2) \log 2$** .

4. Evaluate $\int_0^2 (x^2 + 1) dx$ as the limit of a sum.

Formula: $\int_a^b f(x) dx = \lim_{h \rightarrow 0} h[f(a) + f(a + h) + f(a + 2h) + (\text{and so on up to}) + f(a + (n - 1)h)]$, where $nh = b - a$.

Here $a = 0$, $b = 2$, $f(x) = x^2 + 1$, $nh = 2$.

Sum $S = h \sum_{k=0}^{n-1} (k^2 h^2 + 1) = h^3 \sum k^2 + nh = h^3 \cdot (n - 1)n(2n - 1)/6 + nh$.

Write $h = 2/n$: $S = (8/n^3) \cdot (n - 1)n(2n - 1)/6 + 2 = (8/6)(1 - 1/n)(2 - 1/n) + 2$.

As $n \rightarrow \infty$ ($h \rightarrow 0$): $S \rightarrow (8/6)(1)(2) + 2 = 8/3 + 2 =$ **$14/3$** .

Check by direct integration: $[x^3/3 + x]_0^2 = 8/3 + 2 = 14/3$. ✓

5. Evaluate $\int \sqrt{x^2 + 2x + 5} \, dx$.

Complete the square: $x^2 + 2x + 5 = (x + 1)^2 + 2^2$. Put $X = x + 1$, $a = 2$.

Use $\int \sqrt{X^2 + a^2} \, dX = (X/2)\sqrt{X^2 + a^2} + (a^2/2) \log|X + \sqrt{X^2 + a^2}| + C$.

Answer: $((x + 1)/2)\sqrt{x^2 + 2x + 5} + 2 \log|x + 1 + \sqrt{x^2 + 2x + 5}| + C$.

6. Evaluate $\int e^x(1 + \sin x)/(1 + \cos x) \, dx$.

Write $1 + \sin x = 1 + 2 \sin(x/2)\cos(x/2)$ and $1 + \cos x = 2\cos^2(x/2)$.

$(1 + \sin x)/(1 + \cos x) = 1/(2\cos^2(x/2)) + 2 \sin(x/2)\cos(x/2)/(2\cos^2(x/2)) = (1/2)\sec^2(x/2) + \tan(x/2)$.

Let $f(x) = \tan(x/2)$; then $f'(x) = (1/2)\sec^2(x/2)$. So the integrand is $e^x[f(x) + f'(x)]$.

Answer: $e^x \tan(x/2) + C$.

QB-8. Application of Integrals (Area under Curves)

Key points

- Area between $y = f(x)$, the x -axis and the lines $x = a$, $x = b$ is $A = \int_a^b |f(x)| \, dx$.
- If the curve goes below the x -axis, that part of the integral is negative; take its absolute value and add separately.
- Area between $x = g(y)$, the y -axis and $y = c$, $y = d$ is $\int_c^d |g(y)| \, dy$.
- Area between two curves $y = f(x)$ (upper) and $y = g(x)$ (lower) from $x = a$ to $x = b$ is $\int_a^b [f(x) - g(x)] \, dx$.
- Always draw a rough sketch, find the points of intersection, and use symmetry where possible.

READY RESULTS

Circle $x^2 + y^2 = a^2$: area πa^2 . Ellipse $x^2/a^2 + y^2/b^2 = 1$: area πab . Parabolas $y^2 = 4ax$ and $x^2 = 4ay$ enclose area $16a^2/3$. Parabola $y^2 = 4ax$ and its latus rectum $x = a$ enclose $8a^2/3$.

Never write a negative area. If $\int$ gives a negative number, the region is below the axis — take the modulus.

A. Multiple choice questions

1. Area bounded by $y = x^2$, the x -axis, $x = 0$ and $x = 3$ is (a) 3 (b) 9 (c) 27 (d) 6

(b) $[x^3/3]_0^3 = 9$.

2. Area of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is (a) πa^2 (b) πab (c) $\pi(a + b)$ (d) $2\pi ab$
(b).

3. Area between $y = x$ and $y = x^2$ is (a) $1/2$ (b) $1/3$ (c) $1/6$ (d) 1

(c) $\int_0^1 (x - x^2) \, dx = 1/2 - 1/3 = 1/6$.

4. Area under $y = \sin x$ from $x = 0$ to $x = \pi$ is (a) 1 (b) 2 (c) 0 (d) π

(b) $[-\cos x]_0^\pi = 1 + 1 = 2.$

5. Area bounded by $y = |x|$, the x-axis, $x = -1$ and $x = 1$ is (a) 0 (b) $1/2$ (c) 1 (d) 2

(c) Two triangles each of area $1/2$.

6. Area bounded by $y^2 = 4x$ and the line $x = 1$ is (a) $4/3$ (b) $8/3$ (c) 2 (d) $16/3$

(b) $2 \int_0^1 2\sqrt{x} \, dx = 8/3.$

7. Area under $y = e^x$ from $x = 0$ to $x = 1$ is (a) e (b) $e - 1$ (c) 1 (d) $e + 1$

(b).

8. Area of the circle $x^2 + y^2 = 9$ is (a) 3π (b) 6π (c) 9π (d) 81π

(c).

B. Short answer questions

1. Find the area bounded by $y^2 = x$ and the line $x = 4$.

The region is symmetric about the x-axis. Area = $2 \int_0^4 \sqrt{x} \, dx = 2 \cdot (2/3)[x^{3/2}]_0^4 = (4/3)(8) = \mathbf{32/3 \text{ sq units.}}$

2. Find the area bounded by $y = x^3$, the x-axis and the lines $x = -2$ and $x = 1$.

From -2 to 0 the curve is below the axis: $\int_{-2}^0 x^3 \, dx = -4$, so area = 4. From 0 to 1 : $\int_0^1 x^3 \, dx = 1/4$. Total = $\mathbf{17/4 \text{ sq units.}}$

3. Find the area enclosed between $y = x^2$ and the line $y = 4$.

Intersection: $x^2 = 4 \Rightarrow x = \pm 2$. Area = $\int_{-2}^2 (4 - x^2) \, dx = 16 - 16/3 = \mathbf{32/3 \text{ sq units.}}$

4. Find the area of the ellipse $x^2/16 + y^2/9 = 1$.

$a = 4$, $b = 3$. Area = $4 \int_0^4 (3/4)\sqrt{16 - x^2} \, dx = 3[(x/2)\sqrt{16 - x^2} + 8 \sin^{-1}(x/4)]_0^4 = 3(8 \cdot \pi/2) = \mathbf{12\pi \text{ sq units}} (= \pi ab).$

5. Find the area under $y = \cos x$ from $x = 0$ to $x = \pi/2$.

$[\sin x]_0^{\pi/2} = \mathbf{1 \text{ sq unit.}}$

6. Find the area bounded by $y = 3x + 2$, the x-axis and the lines $x = -1$ and $x = 1$.

The line meets the x-axis at $x = -2/3$. For $-1 \leq x \leq -2/3$ it is below the axis.

$$A_1 = \left| \int_{-1}^{-2/3} (3x + 2) \, dx \right| = \left| [3x^2/2 + 2x]_{-1}^{-2/3} \right| = \left| (2/3 - 4/3) - (3/2 - 2) \right| = \left| -2/3 + 1/2 \right| = 1/6.$$

$$A_2 = \int_{-2/3}^1 (3x + 2) \, dx = (3/2 + 2) - (-2/3) = 7/2 + 2/3 = 25/6.$$

Total = $1/6 + 25/6 = \mathbf{13/3 \text{ sq units.}}$

7. Find the area of the circle $x^2 + y^2 = 4$ lying in the first quadrant.

$$\int_0^2 \sqrt{4 - x^2} dx = [(x/2)\sqrt{4 - x^2} + 2 \sin^{-1}(x/2)]_0^2 = 2 \cdot (\pi/2) = \pi \text{ sq units.}$$

8. Find the area bounded by the curve $x = y^2$, the y-axis and the lines $y = 1$ and $y = 2$.

$$\text{Area} = \int_1^2 y^2 dy = (8 - 1)/3 = 7/3 \text{ sq units.}$$

C. Long answer questions

1. Find the area enclosed between the parabola $y^2 = 4x$ and the line $y = 2x$.

Intersection: $(2x)^2 = 4x \Rightarrow 4x^2 - 4x = 0 \Rightarrow x = 0$ or 1 . Points $(0, 0)$ and $(1, 2)$.

Which is upper? At $x = 1/4$: parabola $y = 2\sqrt{x} = 1$, line $y = 1/2$. So the parabola (upper branch) is above.

$$\text{Area} = \int_0^1 (2\sqrt{x} - 2x) dx = [(4/3)x^{3/2} - x^2]_0^1 = 4/3 - 1 = 1/3 \text{ sq unit.}$$

2. Find the area of the smaller part of the circle $x^2 + y^2 = a^2$ cut off by the line $x = a/\sqrt{2}$.

The smaller part lies between $x = a/\sqrt{2}$ and $x = a$, symmetric about the x-axis.

$$\text{Area} = 2 \int_{a/\sqrt{2}}^a \sqrt{a^2 - x^2} dx = 2[(x/2)\sqrt{a^2 - x^2} + (a^2/2) \sin^{-1}(x/a)]_{a/\sqrt{2}}^a.$$

$$\text{At } x = a: 0 + (a^2/2)(\pi/2) = \pi a^2/4.$$

$$\text{At } x = a/\sqrt{2}: (a/(2\sqrt{2}))(a/\sqrt{2}) + (a^2/2)(\pi/4) = a^2/4 + \pi a^2/8.$$

$$\text{Area} = 2(\pi a^2/4 - a^2/4 - \pi a^2/8) = 2(\pi a^2/8 - a^2/4) = (a^2/4)(\pi - 2) \text{ sq units.}$$

3. Using integration, find the area of the triangle with vertices A(1, 0), B(2, 2) and C(3, 1).

Equations of sides: AB: slope 2, $y = 2(x - 1)$. BC: slope -1 , $y = 4 - x$. AC: slope $1/2$, $y = (x - 1)/2$.

For $1 \leq x \leq 2$ the region lies between AB (top) and AC (bottom); for $2 \leq x \leq 3$ it lies between BC (top) and AC (bottom).

$$\begin{aligned} \text{Area} &= \int_1^2 [2(x - 1) - (x - 1)/2] dx + \int_2^3 [(4 - x) - (x - 1)/2] dx \\ &= \int_1^2 (3/2)(x - 1) dx + \int_2^3 (9/2 - 3x/2) dx = 3/4 + [9x/2 - 3x^2/4]_2^3 = 3/4 + (9/2 - 15/4) = 3/4 + 3/4 = 3/2 \text{ sq units.} \end{aligned}$$

Check with the formula $(1/2)|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)| = (1/2)|1 + 2 - 6| = 3/2 \checkmark$.

4. Find the area of the region $\{(x, y): x^2 \leq y \leq |x|\}$.

$y = |x|$ and $y = x^2$ meet at $x = 0, \pm 1$. The region is symmetric about the y-axis.

$$\text{Area} = 2 \int_0^1 (x - x^2) dx = 2(1/2 - 1/3) = 2(1/6) = 1/3 \text{ sq unit.}$$

5. Find the area bounded by $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$.

Since $x^2 - x + 2 > 0$ always, the parabola lies above the line.

$$\text{Area} = \int_0^3 (x^2 + 2 - x) dx = [x^3/3 + 2x - x^2/2]_0^3 = 9 + 6 - 9/2 = \mathbf{21/2 \text{ sq units.}}$$

6. Find the area enclosed by the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ ($a > 0$).

Intersection: $y = x^2/4a \Rightarrow x^4/16a^2 = 4ax \Rightarrow x(x^3 - 64a^3) = 0 \Rightarrow x = 0$ or $x = 4a$. Points $(0, 0)$ and $(4a, 4a)$.

Upper curve: $y = 2\sqrt{ax}$; lower: $y = x^2/4a$.

$$\begin{aligned} \text{Area} &= \int_0^{4a} [2\sqrt{a} x^{1/2} - x^2/4a] dx = [(4\sqrt{a}/3)x^{3/2} - x^3/12a]_0^{4a} = (4\sqrt{a}/3)(8a\sqrt{a}) - \\ &64a^3/12a = 32a^2/3 - 16a^2/3 = \mathbf{16a^2/3 \text{ sq units.}} \end{aligned}$$

QB-9. Differential Equations

Key points

- A differential equation (DE) is an equation containing derivatives such as dy/dx , d^2y/dx^2 .
- **Order** = order of the highest derivative present. **Degree** = power of that highest derivative, after the equation is made free of radicals and fractions in the derivatives. Degree is **not defined** if derivatives appear inside $\sin$, $\cos$, $\log$, $e^{(\)}$ etc.
- The **general solution** of an n -th order DE has n arbitrary constants. A **particular solution** is obtained by giving values to these constants using given conditions.
- To **form** a DE from a family of curves with n constants, differentiate n times and eliminate the constants.
- **Variable separable**: write $f(y) dy = g(x) dx$ and integrate both sides.
- **Homogeneous**: $dy/dx = F(y/x)$. Put $y = vx$, $dy/dx = v + x dv/dx$; then separate v and x .
- **Linear**: $dy/dx + Py = Q$ (P, Q functions of x). Integrating factor $IF = e^{\int P dx}$. Solution: $y \cdot IF = \int Q \cdot IF dx + C$.
- Linear in x : $dx/dy + P_1x = Q_1$ (functions of y). $IF = e^{\int P_1 dy}$; $x \cdot IF = \int Q_1 \cdot IF dy + C$.

QUICK TEST FOR HOMOGENEOUS

Replace x by λx and y by λy in the right side of $dy/dx = f(x, y)$. If λ cancels completely, the equation is homogeneous.

USEFUL INTEGRATING FACTORS

$P = 1/x \Rightarrow IF = x$. $P = -1/x \Rightarrow IF = 1/x$. $P = 2/x \Rightarrow IF = x^2$. $P = \tan x \Rightarrow IF = \sec x$. $P = \cot x \Rightarrow IF = \sin x$. $P = -\tan x \Rightarrow IF = \cos x$. $P = k$ (constant) $\Rightarrow IF = e^{kx}$.

A. Multiple choice questions

1. The order and degree of $(d^2y/dx^2)^3 + (dy/dx)^4 + y = 0$ are (a) 2, 4 (b) 2, 3 (c) 4, 2 (d) 3, 2
(b) Highest derivative is second order; its power is 3.
2. The degree of $d^2y/dx^2 + \sin(dy/dx) = 0$ is (a) 1 (b) 2 (c) 0 (d) not defined
(d) dy/dx appears inside sine.
3. The integrating factor of $dy/dx + y/x = x^2$ is (a) x (b) $1/x$ (c) e^x (d) $\log x$
(a) $e^{\int dx/x} = e^{\log x} = x$.
4. The general solution of $dy/dx = y$ is (a) $y = x + C$ (b) $y = Ce^x$ (c) $y = e^x + C$ (d) $y = Cx$
(b).
5. The number of arbitrary constants in the general solution of a third-order DE is (a) 0 (b) 1 (c) 2 (d) 3
(d).

6. The integrating factor of $dy/dx - y \tan x = \cos x$ is (a) $\sec x$ (b) $\cos x$ (c) $\tan x$ (d) $e^{\tan x}$

(b) $e^{-\int \tan x \, dx} = e^{\log \cos x} = \cos x.$

7. Which equation is homogeneous? (a) $dy/dx = (x + y)/x$ (b) $dy/dx = x + y^2$ (c) $dy/dx = x^2 + 1$ (d) $dy/dx = e^x + y$

(a).

8. The solution of $dy/dx = e^x + y$ is (a) $e^x + e^y = C$ (b) $e^x + e^{-y} = C$ (c) $e^x - e^y = C$ (d) $e^{-x} + e^y = C$

(b) $e^{-y} dy = e^x dx \Rightarrow -e^{-y} = e^x - C.$

9. The order of the DE of the family $y = A \cos x + B \sin x$ is (a) 1 (b) 2 (c) 3 (d) 0

(b) Two constants $\Rightarrow$ order 2 (the DE is $y'' + y = 0$).

10. The DE whose general solution is $y = Ce^x$ is (a) $y' = x$ (b) $y' = y$ (c) $y' = 1$ (d) $y' = -y$
(b).

B. Short answer questions

1. Solve $dy/dx = (1 + y^2)/(1 + x^2)$.

Separate: $dy/(1 + y^2) = dx/(1 + x^2)$. Integrate: $\tan^{-1}y = \tan^{-1}x + C.$

2. Solve $x \, dy - y \, dx = 0$.

$dy/y = dx/x \Rightarrow \log|y| = \log|x| + \log C \Rightarrow y = Cx$ (straight lines through the origin).

3. Solve $dy/dx + 2y = e^{-x}$.

$P = 2, Q = e^{-x}$. IF $= e^{2x}$. $y e^{2x} = \int e^{-x} e^{2x} dx = e^x + C \Rightarrow y = e^{-x} + C e^{-2x}.$

4. Form the DE of the family $y = a e^x + b e^{-x}$.

$y' = a e^x - b e^{-x}$; $y'' = a e^x + b e^{-x} = y$. DE: $d^2y/dx^2 - y = 0.$

5. Solve $dy/dx = x e^x$.

$y = \int x e^x dx = x e^x - e^x + C$, i.e. $y = (x - 1)e^x + C.$

6. Find the particular solution of $dy/dx = -4xy^2$, given $y = 1$ when $x = 0$.

$-dy/y^2 = 4x \, dx \Rightarrow 1/y = 2x^2 + C$. At $x = 0, y = 1$: $C = 1$. So $y = 1/(1 + 2x^2).$

7. Write the order and degree of $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$.

Highest derivative y''' , its power 2. **Order 3, degree 2.**

8. Solve $(x + 1) dy/dx = 2xy$.

$dy/y = 2x/(x + 1) dx = [2 - 2/(x + 1)] dx$. $\log|y| = 2x - 2 \log|x + 1| + C \Rightarrow y(x + 1)^2 = A e^{2x}.$

9. Solve $dy/dx + y = 1$.

IF = e^x ; $y e^x = \int e^x dx = e^x + C \Rightarrow y = 1 + C e^{-x}$. (It can also be separated: $dy/(1 - y) = dx$.)

C. Long answer questions**1. Solve the homogeneous equation $(x - y) dy - (x + y) dx = 0$.**

$dy/dx = (x + y)/(x - y)$. Check: replacing x, y by $\lambda x, \lambda y$ leaves it unchanged, so it is homogeneous.

Put $y = vx$: $v + x dv/dx = (1 + v)/(1 - v)$.

$$x dv/dx = (1 + v)/(1 - v) - v = (1 + v - v + v^2)/(1 - v) = (1 + v^2)/(1 - v).$$

Separate: $(1 - v)/(1 + v^2) dv = dx/x$.

Integrate: $\int dv/(1 + v^2) - \int v/(1 + v^2) dv = \log|x| + C \Rightarrow \tan^{-1}v - (1/2) \log(1 + v^2) = \log|x| + C$.

Put $v = y/x$: $\tan^{-1}(y/x) - (1/2) \log((x^2 + y^2)/x^2) = \log|x| + C$. Since $(1/2) \log(x^2) = \log|x|$, these cancel:

$$\tan^{-1}(y/x) = (1/2) \log(x^2 + y^2) + C.$$

2. Solve $x dy/dx + 2y = x^2$ ($x \neq 0$).

Divide by x : $dy/dx + (2/x)y = x$. $P = 2/x$, $Q = x$.

$$\text{IF} = e^{\int 2/x dx} = e^{2 \log x} = x^2.$$

$$y \cdot x^2 = \int x \cdot x^2 dx = x^4/4 + C.$$

$$y = x^2/4 + C/x^2.$$

3. Find the particular solution of $dy/dx + y \cot x = 2x + x^2 \cot x$, given $y = 0$ when $x = \pi/2$.

$$P = \cot x \Rightarrow \text{IF} = e^{\log \sin x} = \sin x.$$

$$y \sin x = \int (2x + x^2 \cot x) \sin x dx = \int (2x \sin x + x^2 \cos x) dx.$$

Notice $d/dx(x^2 \sin x) = 2x \sin x + x^2 \cos x$. So $y \sin x = x^2 \sin x + C \Rightarrow y = x^2 + C \operatorname{cosec} x$.

$$x = \pi/2, y = 0: 0 = \pi^2/4 + C \Rightarrow C = -\pi^2/4.$$

$$y = x^2 - (\pi^2/4) \operatorname{cosec} x.$$

4. Solve $(1 + e^{2x}) dy + (1 + y^2) e^x dx = 0$, given $y = 1$ when $x = 0$.

$$\text{Separate: } dy/(1 + y^2) + e^x dx/(1 + e^{2x}) = 0.$$

In the second integral put $e^x = t$, $e^x dx = dt$: $\int dt/(1 + t^2) = \tan^{-1}(e^x)$.

$$\text{So } \tan^{-1}y + \tan^{-1}(e^x) = C.$$

$$x = 0, y = 1: \pi/4 + \pi/4 = C \Rightarrow C = \pi/2.$$

$$\tan^{-1}y + \tan^{-1}(e^x) = \pi/2.$$

5. Form the differential equation of the family of circles touching the y-axis at the origin.

Such a circle has centre $(a, 0)$ and radius a : $(x - a)^2 + y^2 = a^2 \Rightarrow x^2 + y^2 = 2ax$. — (1)

Differentiate: $2x + 2y y' = 2a \Rightarrow a = x + y y'$.

Put in (1): $x^2 + y^2 = 2x(x + y y') = 2x^2 + 2xy y'$.

$2xy \, dy/dx + x^2 - y^2 = 0$ (order 1, degree 1).

6. The number of bacteria increases at a rate proportional to the number present. If 1,00,000 bacteria increase by 10% in 2 hours, in how many hours will the count reach 2,00,000?

$$dN/dt = kN \Rightarrow N = N_0 e^{kt}.$$

After 2 hours, $N = 1.1N_0$: $e^{2k} = 1.1 \Rightarrow k = (1/2) \log 1.1$.

For $N = 2N_0$: $e^{kt} = 2 \Rightarrow t = \log 2/k = 2 \log 2 / \log 1.1$.

$t = 2 \times 0.6931/0.0953 \approx \mathbf{14.5 \text{ hours}}$.

QB-10. Vector Algebra

Key points

- A vector has magnitude and direction. If $\mathbf{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, then $|\mathbf{a}| = \sqrt{(a_1^2 + a_2^2 + a_3^2)}$ and the unit vector $\hat{a} = \mathbf{a}/|\mathbf{a}|$.
- Direction cosines: $l = a_1/|\mathbf{a}|$, $m = a_2/|\mathbf{a}|$, $n = a_3/|\mathbf{a}|$, with $l^2 + m^2 + n^2 = 1$.
- Section formula: point dividing $A(\mathbf{a})$ and $B(\mathbf{b})$ in ratio $m : n$ internally is $(m\mathbf{b} + n\mathbf{a})/(m + n)$; externally $(m\mathbf{b} - n\mathbf{a})/(m - n)$.
- **Dot product:** $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta = a_1b_1 + a_2b_2 + a_3b_3$. $\mathbf{a} \perp \mathbf{b} \Leftrightarrow \mathbf{a} \cdot \mathbf{b} = 0$.
- Projection of $\mathbf{a}$ on $\mathbf{b} = (\mathbf{a} \cdot \mathbf{b})/|\mathbf{b}|$.
- **Cross product:** $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin \theta \hat{n}$. Its magnitude is the area of the parallelogram with sides $\mathbf{a}$, $\mathbf{b}$. $\mathbf{a} \parallel \mathbf{b} \Leftrightarrow \mathbf{a} \times \mathbf{b} = \mathbf{0}$.
- Area of triangle $ABC = (1/2)|\mathbf{AB} \times \mathbf{AC}|$.
- **Scalar triple product** $[\mathbf{a} \, \mathbf{b} \, \mathbf{c}] = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ = the 3×3 determinant of components = volume of the parallelepiped. $[\mathbf{a} \, \mathbf{b} \, \mathbf{c}] = 0 \Leftrightarrow$ coplanar.

CROSS PRODUCT AS A DETERMINANT

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\hat{i} - (a_1b_3 - a_3b_1)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}.$$

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1; \hat{i} \cdot \hat{j} = 0. \hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}; \hat{i} \times \hat{i} = \mathbf{0}.$$

$$\text{Lagrange identity: } |\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = |\mathbf{a}|^2 |\mathbf{b}|^2.$$

$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$: the cross product is NOT commutative, but the dot product is.

A. Multiple choice questions

1. If $\mathbf{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, then $|\mathbf{a}| =$ (a) 4 (b) $\sqrt{14}$ (c) 14 (d) $\sqrt{12}$

(b) $\sqrt{4 + 9 + 1}$.

2. $\hat{i} \cdot (\hat{j} \times \hat{k}) =$ (a) 0 (b) 1 (c) -1 (d) 3

(b) $\hat{j} \times \hat{k} = \hat{i}$ and $\hat{i} \cdot \hat{i} = 1$.

3. Two non-zero vectors are perpendicular if (a) $\mathbf{a} \times \mathbf{b} = 0$ (b) $\mathbf{a} \cdot \mathbf{b} = 0$ (c) $\mathbf{a} = \mathbf{b}$ (d) $|\mathbf{a}| = |\mathbf{b}|$

(b).

4. The unit vector along $\hat{i} + \hat{j}$ is (a) $\hat{i} + \hat{j}$ (b) $(\hat{i} + \hat{j})/2$ (c) $(\hat{i} + \hat{j})/\sqrt{2}$ (d) $\sqrt{2}(\hat{i} + \hat{j})$

(c).

5. If $|\mathbf{a}| = |\mathbf{b}| = 1$ and $\mathbf{a} \cdot \mathbf{b} = 1/2$, the angle between them is (a) 30° (b) 45° (c) 60° (d) 90°

(c) $\cos \theta = 1/2$.

6. The area of the parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is (a) $\mathbf{a} \cdot \mathbf{b}$ (b) $|\mathbf{a} \times \mathbf{b}|$ (c) $(1/2)|\mathbf{a} \times \mathbf{b}|$ (d) $|\mathbf{a}||\mathbf{b}|$

(b).

7. $2\hat{i} + \lambda\hat{j} + \hat{k}$ is perpendicular to $\hat{i} - 2\hat{j} + 3\hat{k}$ if $\lambda =$ (a) $5/2$ (b) 2 (c) $-5/2$ (d) 3

(a) $2 - 2\lambda + 3 = 0$.

8. $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = 0$ means $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are (a) collinear (b) coplanar (c) mutually perpendicular (d) unit vectors

(b).

9. The projection of $\hat{i} + \hat{j} + \hat{k}$ on $\hat{j}$ is (a) 0 (b) 1 (c) $\sqrt{3}$ (d) 3

(b).

10. $|\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 =$ (a) 0 (b) $|\mathbf{a}|^2|\mathbf{b}|^2$ (c) $|\mathbf{a}||\mathbf{b}|$ (d) 1

(b) $\sin^2 \theta + \cos^2 \theta = 1$.

B. Short answer questions

1. Find the angle between $\mathbf{a} = \hat{i} + \hat{j} - \hat{k}$ and $\mathbf{b} = \hat{i} - \hat{j} + \hat{k}$.

$\mathbf{a} \cdot \mathbf{b} = 1 - 1 - 1 = -1$; $|\mathbf{a}| = |\mathbf{b}| = \sqrt{3}$. $\cos \theta = -1/3 \Rightarrow \theta = \cos^{-1}(-1/3)$.

2. Find a unit vector perpendicular to both $\mathbf{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\mathbf{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$.

$\mathbf{a} \times \mathbf{b} = (1 \cdot (-2) - 3 \cdot 5)\hat{i} - (2 \cdot (-2) - 3 \cdot 3)\hat{j} + (2 \cdot 5 - 1 \cdot 3)\hat{k} = -17\hat{i} + 13\hat{j} + 7\hat{k}$.

$|\mathbf{a} \times \mathbf{b}| = \sqrt{(289 + 169 + 49)} = \sqrt{507}$. Unit vector = $\pm(-17\hat{i} + 13\hat{j} + 7\hat{k})/\sqrt{507}$.

3. Find the projection of $\mathbf{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on $\mathbf{b} = \hat{i} + 2\hat{j} + \hat{k}$.

$\mathbf{a} \cdot \mathbf{b} = 2 + 6 + 2 = 10$; $|\mathbf{b}| = \sqrt{6}$. Projection = $10/\sqrt{6} = 5\sqrt{6}/3$.

4. Show that $A(1, 2, 7)$, $B(2, 6, 3)$, $C(3, 10, -1)$ are collinear.

$\overrightarrow{AB} = \hat{i} + 4\hat{j} - 4\hat{k}$; $\overrightarrow{BC} = \hat{i} + 4\hat{j} - 4\hat{k}$. $\overrightarrow{AB} = \overrightarrow{BC}$ and B is common $\Rightarrow A, B, C$ are collinear.

5. Find the area of the triangle with vertices $A(1, 1, 1)$, $B(1, 2, 3)$, $C(2, 3, 1)$.

$\overrightarrow{AB} = \hat{j} + 2\hat{k}$, $\overrightarrow{AC} = \hat{i} + 2\hat{j}$. $\overrightarrow{AB} \times \overrightarrow{AC} = (1 \cdot 0 - 2 \cdot 2)\hat{i} - (0 \cdot 0 - 2 \cdot 1)\hat{j} + (0 \cdot 2 - 1 \cdot 1)\hat{k} = -4\hat{i} + 2\hat{j} - \hat{k}$. Magnitude $\sqrt{21}$. Area = $\sqrt{21}/2$ sq units.

6. If $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$ and $\mathbf{a} \perp \mathbf{b}$, find $|\mathbf{a} - \mathbf{b}|$.

$$|\mathbf{a} - \mathbf{b}|^2 = 9 + 16 - 2(0) = 25 \Rightarrow \mathbf{5}.$$

7. Find λ if $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + \lambda\hat{j} + 5\hat{k}$ are coplanar.

$$\text{Determinant} = 2(10 + 3\lambda) + 1(5 + 9) + 1(\lambda - 6) = 28 + 7\lambda = 0 \Rightarrow \lambda = -4.$$

8. Find the direction cosines of $\hat{i} + 2\hat{j} + 3\hat{k}$.

$$\text{Magnitude } \sqrt{14}. \text{ DCs: } \mathbf{1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14}}.$$

9. P and Q have position vectors $2\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - 3\mathbf{b}$. Find the point dividing PQ internally in the ratio 1 : 2.

$$[1(\mathbf{a} - 3\mathbf{b}) + 2(2\mathbf{a} + \mathbf{b})]/3 = \mathbf{(5a - b)/3}.$$

C. Long answer questions

1. If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$, $|\mathbf{c}| = 7$, find the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$\mathbf{c} = -(\mathbf{a} + \mathbf{b}) \Rightarrow |\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}.$$

$$49 = 9 + 25 + 2\mathbf{a} \cdot \mathbf{b} \Rightarrow \mathbf{a} \cdot \mathbf{b} = 15/2.$$

$$\cos \theta = (15/2)/(3 \times 5) = 1/2 \Rightarrow \mathbf{\theta = 60^\circ}.$$

2. Find the volume of the parallelepiped whose edges are $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}$, $\mathbf{b} = \hat{i} - \hat{j} + \hat{k}$, $\mathbf{c} = \hat{i} + 2\hat{j} - \hat{k}$.

$$[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 1(1 - 2) - 1(-1 - 1) + 1(2 + 1) = -1 + 2 + 3 = 4.$$

$$\text{Volume} = \mathbf{4 \text{ cubic units}}.$$

3. Prove by vectors that the angle in a semicircle is a right angle.

Let O be the centre, AB a diameter and P any point on the circle. Take O as origin: $\mathbf{OA} = -\mathbf{a}$, $\mathbf{OB} = \mathbf{a}$, $\mathbf{OP} = \mathbf{r}$, with $|\mathbf{r}| = |\mathbf{a}|$ (radius).

$$\mathbf{AP} = \mathbf{r} + \mathbf{a}, \mathbf{BP} = \mathbf{r} - \mathbf{a}.$$

$$\mathbf{AP} \cdot \mathbf{BP} = (\mathbf{r} + \mathbf{a}) \cdot (\mathbf{r} - \mathbf{a}) = |\mathbf{r}|^2 - |\mathbf{a}|^2 = 0.$$

$$\text{So } \mathbf{AP} \perp \mathbf{BP}, \text{ i.e. } \mathbf{\angle APB = 90^\circ}.$$

4. Let $\mathbf{a} = \hat{i} + 4\hat{j} + 2\hat{k}$, $\mathbf{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$, $\mathbf{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find a vector $\mathbf{d}$ perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ with $\mathbf{c} \cdot \mathbf{d} = 15$.

$$\mathbf{d} \text{ is parallel to } \mathbf{a} \times \mathbf{b} = (4 \cdot 7 - 2 \cdot (-2))\hat{i} - (1 \cdot 7 - 2 \cdot 3)\hat{j} + (1 \cdot (-2) - 4 \cdot 3)\hat{k} = 32\hat{i} - \hat{j} - 14\hat{k}.$$

$$\text{Let } \mathbf{d} = \lambda(32\hat{i} - \hat{j} - 14\hat{k}). \mathbf{c} \cdot \mathbf{d} = \lambda(64 + 1 - 56) = 9\lambda = 15 \Rightarrow \lambda = 5/3.$$

$$\mathbf{d} = \mathbf{(1/3)(160\hat{i} - 5\hat{j} - 70\hat{k})}.$$

5. Prove that $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$ and interpret it.

$$(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = \mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{b} - \mathbf{b} \times \mathbf{a} - \mathbf{b} \times \mathbf{b} = \mathbf{0} + \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{b} - \mathbf{0} = 2(\mathbf{a} \times \mathbf{b}).$$

Meaning: $\mathbf{a} + \mathbf{b}$ and $\mathbf{a} - \mathbf{b}$ are the diagonals of the parallelogram with sides $\mathbf{a}$, $\mathbf{b}$. So the area of a parallelogram $= (1/2)|\mathbf{d}_1 \times \mathbf{d}_2|$, where $\mathbf{d}_1, \mathbf{d}_2$ are its diagonals.

QB-11. Three-Dimensional Geometry

Key points

- Direction cosines (DCs) l, m, n of a line satisfy $l^2 + m^2 + n^2 = 1$. Direction ratios (DRs) a, b, c are any numbers proportional to l, m, n .
- DRs of the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) : $x_2 - x_1, y_2 - y_1, z_2 - z_1$.
- Line through point **a** parallel to **b**: $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$; Cartesian: $(x - x_1)/a = (y - y_1)/b = (z - z_1)/c$.
- Angle between two lines: $\cos \theta = |a_1 a_2 + b_1 b_2 + c_1 c_2| / (\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2})$.
- Plane with normal (a, b, c) through (x_1, y_1, z_1) : $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$. Vector form: $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$.
- Intercept form: $x/a + y/b + z/c = 1$. Normal form: $lx + my + nz = p$ (p = distance from origin).
- Plane through the intersection of two planes: $(a_1 x + b_1 y + c_1 z + d_1) + \lambda(a_2 x + b_2 y + c_2 z + d_2) = 0$.
- Angle between line (DR **b**) and plane (normal **n**): $\sin \phi = |\mathbf{b} \cdot \mathbf{n}| / (|\mathbf{b}| |\mathbf{n}|)$.

DISTANCE FORMULAS

Point (x_1, y_1, z_1) to plane $ax + by + cz + d = 0$: $|ax_1 + by_1 + cz_1 + d| / \sqrt{a^2 + b^2 + c^2}$.

Between parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$: $|d_1 - d_2| / \sqrt{a^2 + b^2 + c^2}$.

Shortest distance between skew lines $\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}_2$: $d = |(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)| / |\mathbf{b}_1 \times \mathbf{b}_2|$.

Between parallel lines $\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}$ and $\mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}$: $d = |\mathbf{b} \times (\mathbf{a}_2 - \mathbf{a}_1)| / |\mathbf{b}|$.

LINES INTERSECT?

If the shortest distance comes out as 0, the two lines intersect (they are coplanar).

A. Multiple choice questions

1. The direction cosines of the x-axis are (a) 0, 0, 1 (b) 1, 0, 0 (c) 1, 1, 1 (d) 0, 1, 0
(b).

2. The distance of (2, 3, 4) from the plane $3x - 6y + 2z + 11 = 0$ is (a) 1 (b) 2 (c) 3 (d) 7
(a) $|6 - 18 + 8 + 11| / \sqrt{9 + 36 + 4} = 7/7 = 1$.

3. The angle between the planes $2x - y + z = 6$ and $x + y + 2z = 7$ is (a) 30° (b) 45° (c) 60° (d) 90°
(c) $\cos \theta = (2 - 1 + 2) / (\sqrt{6} \cdot \sqrt{6}) = 1/2$.

4. The plane with intercepts 2, 3, 4 on the axes is (a) $2x + 3y + 4z = 1$ (b) $x/2 + y/3 + z/4 = 1$ (c) $6x + 4y + 3z = 1$ (d) $x + y + z = 9$
(b).

- 5. The distance of the plane $2x - 3y + 6z = 7$ from the origin is (a) 7 (b) 1 (c) $7/\sqrt{7}$ (d) 2**
 (b) $\sqrt{4 + 9 + 36} = 7$, so distance $7/7 = 1$.
- 6. The DRs of the line $(x - 2)/3 = (y + 1)/(-2) = z/5$ are (a) 2, -1, 0 (b) 3, -2, 5 (c) 3, 2, 5 (d) -2, 1, 0**
 (b).
- 7. A line with DR $\mathbf{b}$ is parallel to a plane with normal $\mathbf{n}$ if (a) $\mathbf{b} \times \mathbf{n} = \mathbf{0}$ (b) $\mathbf{b} \cdot \mathbf{n} = 0$ (c) $\mathbf{b} = \mathbf{n}$ (d) $|\mathbf{b}| = |\mathbf{n}|$**
 (b) The line is perpendicular to the normal.
- 8. The plane through the origin perpendicular to the vector $\hat{i} + 2\hat{j} + 3\hat{k}$ is (a) $x + 2y + 3z = 1$ (b) $x + 2y + 3z = 0$ (c) $3x + 2y + z = 0$ (d) $x + y + z = 6$**
 (b).
- 9. Lines with DRs 1, -2, 1 and 2, 1, 0 are (a) parallel (b) perpendicular (c) inclined at 45° (d) coincident**
 (b) $2 - 2 + 0 = 0$.
- 10. If l, m, n are DCs of a line, $l^2 + m^2 + n^2 =$ (a) 0 (b) 1 (c) 2 (d) 3**
 (b).

B. Short answer questions

1. Find the angle between the lines $\mathbf{r} = 3\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})$ and $\mathbf{r} = 5\hat{i} - 2\hat{j} + \mu(3\hat{i} + 2\hat{j} + 6\hat{k})$.

$$\cos \theta = (3 + 4 + 12)/(3 \times 7) = 19/21 \Rightarrow \theta = \cos^{-1}(19/21).$$

2. Find the Cartesian equation of the line through (1, 2, 3) parallel to $3\hat{i} + 2\hat{j} - 2\hat{k}$.
 $(x - 1)/3 = (y - 2)/2 = (z - 3)/(-2).$

3. Find the distance between the parallel planes $2x - y + 2z + 3 = 0$ and $4x - 2y + 4z + 5 = 0$.

Write the second as $2x - y + 2z + 5/2 = 0$. Distance = $|3 - 5/2|/\sqrt{4 + 1 + 4} = (1/2)/3 = 1/6$.

4. Find the equation of the plane through (1, 0, -2) with normal $\hat{i} + \hat{j} - \hat{k}$.

$$1(x - 1) + 1(y - 0) - 1(z + 2) = 0 \Rightarrow x + y - z = 3.$$

5. Find the foot of the perpendicular from the origin to the plane $2x - 3y + 4z - 6 = 0$.

The foot is $(2t, -3t, 4t)$ for some t . Putting in the plane: $4t + 9t + 16t = 6 \Rightarrow t = 6/29$.
 Foot = $(12/29, -18/29, 24/29)$.

6. Find the angle between the line $(x + 1)/2 = y/3 = (z - 3)/6$ and the plane $10x + 2y - 11z = 3$.

$$\mathbf{b} = (2, 3, 6), |\mathbf{b}| = 7; \mathbf{n} = (10, 2, -11), |\mathbf{n}| = 15. \sin \phi = |20 + 6 - 66|/105 = 40/105 = 8/21 \Rightarrow \phi = \sin^{-1}(8/21).$$

7. Find k if the lines $(x - 1)/(-3) = (y - 2)/(2k) = (z - 3)/2$ and $(x - 1)/(3k) = (y - 1)/1 = (z - 6)/(-5)$ are perpendicular.

$$(-3)(3k) + (2k)(1) + (2)(-5) = 0 \Rightarrow -7k = 10 \Rightarrow \mathbf{k = -10/7}.$$

8. Find the plane through $(1, 1, 0)$, $(1, 2, 1)$ and $(-2, 2, -1)$.

Vectors in the plane: $(0, 1, 1)$ and $(-3, 1, -1)$. Normal = $(0, 1, 1) \times (-3, 1, -1) = (1 \cdot (-1) - 1 \cdot 1, -(0 \cdot (-1) - 1 \cdot (-3)), 0 \cdot 1 - 1 \cdot (-3)) = (-2, -3, 3)$.

$$-2(x - 1) - 3(y - 1) + 3z = 0 \Rightarrow \mathbf{2x + 3y - 3z = 5}.$$
 (Check $(-2, 2, -1)$: $-4 + 6 + 3 = 5$ ✓.)

C. Long answer questions

1. Find the shortest distance between $\mathbf{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\mathbf{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$.

$$\mathbf{a_1} = (1, 2, 1), \mathbf{b_1} = (1, -1, 1); \mathbf{a_2} = (2, -1, -1), \mathbf{b_2} = (2, 1, 2).$$

$$\mathbf{a_2} - \mathbf{a_1} = (1, -3, -2).$$

$$\mathbf{b_1} \times \mathbf{b_2} = ((-1)(2) - (1)(1), -((1)(2) - (1)(2)), (1)(1) - (-1)(2)) = (-3, 0, 3). |\mathbf{b_1} \times \mathbf{b_2}| = \sqrt{18} = 3\sqrt{2}.$$

$$(\mathbf{a_2} - \mathbf{a_1}) \cdot (\mathbf{b_1} \times \mathbf{b_2}) = -3 + 0 - 6 = -9.$$

$$d = |-9|/(3\sqrt{2}) = \mathbf{3/\sqrt{2} = 3\sqrt{2}/2 \text{ units}}.$$

2. Find the shortest distance between $(x + 1)/7 = (y + 1)/(-6) = (z + 1)/1$ and $(x - 3)/1 = (y - 5)/(-2) = (z - 7)/1$.

Points: $A_1(-1, -1, -1)$, $A_2(3, 5, 7)$; $A_2 - A_1 = (4, 6, 8)$.

$$\mathbf{b_1} \times \mathbf{b_2} = (7, -6, 1) \times (1, -2, 1) = ((-6)(1) - (1)(-2), -(7 \cdot 1 - 1 \cdot 1), 7(-2) - (-6)(1)) = (-4, -6, -8).$$

$$\text{Magnitude} = \sqrt{16 + 36 + 64} = \sqrt{116} = 2\sqrt{29}. \text{ Dot product} = -16 - 36 - 64 = -116.$$

$$d = 116/(2\sqrt{29}) = 58/\sqrt{29} = \mathbf{2\sqrt{29} \text{ units}}.$$

3. Find the plane through the line of intersection of $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to $x - y + z = 0$.

Required plane: $(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0$, normal $(1 + 2\lambda, 1 + 3\lambda, 1 + 4\lambda)$.

$$\text{Perpendicular to } (1, -1, 1): (1 + 2\lambda) - (1 + 3\lambda) + (1 + 4\lambda) = 0 \Rightarrow 1 + 3\lambda = 0 \Rightarrow \lambda = -1/3.$$

$$\text{Multiply by 3: } 3(x + y + z - 1) - (2x + 3y + 4z - 5) = 0 \Rightarrow \mathbf{x - z + 2 = 0}.$$

4. Find the image of the point (1, 2, 3) in the plane $x + 2y + 4z = 38$.

The perpendicular from P(1, 2, 3) has DRs (1, 2, 4): any point on it is $(1 + t, 2 + 2t, 3 + 4t)$.

Foot N lies on the plane: $(1 + t) + 2(2 + 2t) + 4(3 + 4t) = 38 \Rightarrow 17 + 21t = 38 \Rightarrow t = 1$.
 $N = (2, 4, 7)$.

N is the midpoint of P and its image Q: $Q = 2N - P = (4 - 1, 8 - 2, 14 - 3) = \mathbf{(3, 6, 11)}$.

5. Find the distance of $(-1, -5, -10)$ from the point where the line $\mathbf{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ meets the plane $\mathbf{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$.

General point: $(2 + 3\lambda, -1 + 4\lambda, 2 + 2\lambda)$. In the plane: $(2 + 3\lambda) - (-1 + 4\lambda) + (2 + 2\lambda) = 5 \Rightarrow 5 + \lambda = 5 \Rightarrow \lambda = 0$. Point $(2, -1, 2)$.

Distance $= \sqrt{(2 + 1)^2 + (-1 + 5)^2 + (2 + 10)^2} = \sqrt{9 + 16 + 144} = \sqrt{169} = \mathbf{13 \text{ units}}$.

6. Find the equation of the line through $(1, 2, -4)$ perpendicular to both $(x - 8)/3 = (y + 19)/(-16) = (z - 10)/7$ and $(x - 15)/3 = (y - 29)/8 = (z - 5)/(-5)$.

Required DR $= (3, -16, 7) \times (3, 8, -5) = ((-16)(-5) - 7 \cdot 8, -(3 \cdot (-5) - 7 \cdot 3), 3 \cdot 8 - (-16) \cdot 3) = (24, 36, 72) = 12(2, 3, 6)$.

$\mathbf{(x - 1)/2 = (y - 2)/3 = (z + 4)/6}$; vector form $\mathbf{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$.

QB-12. Linear Programming

Key points

- An LPP asks to maximise or minimise a linear **objective function** $Z = ax + by$ subject to linear **constraints** (inequalities) and **non-negativity** $x \geq 0, y \geq 0$.
- **Feasible region**: the common region satisfying all constraints. Every point in it is a feasible solution.
- **Corner point theorem**: if the feasible region is bounded, the maximum and minimum of Z occur at corner points.
- If the region is unbounded, a value found at a corner is the optimum only if the open half-plane $ax + by > M$ (for max) or $< m$ (for min) has no point in common with the region.
- If two corners give the same optimum, every point on the segment joining them is also optimal.

EXAM STEPS (GRAPHICAL METHOD)

(1) Define variables. (2) Write Z and constraints. (3) Draw each line using its two intercepts and shade the correct side (test with (0, 0)). (4) Find corner points, solving pairs of equations. (5) Make a table of Z at each corner. (6) State the answer in words.

A. Multiple choice questions

1. In an LPP the objective function is always (a) quadratic (b) linear (c) cubic (d) constant

(b).

2. For a bounded feasible region, the optimal value occurs at (a) the origin (b) a corner point (c) the centre (d) any interior point

(b).

3. Maximum of $Z = 3x + 4y$ subject to $x + y \leq 4$, $x, y \geq 0$ is (a) 12 (b) 16 (c) 7 (d) 0

(b) Corners (0, 0), (4, 0), (0, 4) give 0, 12, 16.

4. Minimum of $Z = 2x + 3y$ subject to $x + y \geq 2$, $x, y \geq 0$ is (a) 4 (b) 6 (c) 0 (d) 5

(a) At (2, 0). (Since $2x + 3y \geq 2(x + y) \geq 4$ in the region.)

5. Corner points of a feasible region are (0, 0), (5, 0), (3, 4), (0, 5). Maximum of $Z = 3x + 2y$ is (a) 15 (b) 17 (c) 10 (d) 18

(b) At (3, 4): $9 + 8 = 17$.

6. The constraints $x \geq 0$, $y \geq 0$ are called (a) objective (b) non-negativity constraints (c) optimal (d) infeasible

(b).

7. A point satisfying all the constraints is a (a) feasible solution (b) optimal solution (c) corner point (d) basic point

(a).

8. If Z is maximum at two corner points A and B, then Z is maximum at (a) A only (b) B only (c) every point of segment AB (d) no point

(c).

B. Short answer questions

1. Maximise $Z = 3x + 2y$ subject to $x + 2y \leq 10$, $3x + y \leq 15$, $x, y \geq 0$.

Intersection: $x = 10 - 2y \Rightarrow 30 - 6y + y = 15 \Rightarrow y = 3$, $x = 4$.

Corner	$Z = 3x + 2y$
(0, 0)	0
(5, 0)	15
(4, 3)	18
(0, 5)	10

Max $Z = 18$ at (4, 3).

2. Minimise $Z = -3x + 4y$ subject to $x + 2y \leq 8$, $3x + 2y \leq 12$, $x, y \geq 0$.

Intersection: subtract the equations: $2x = 4 \Rightarrow x = 2$, $y = 3$. Corners: (0, 0) $\rightarrow 0$; (4, 0) $\rightarrow -12$; (2, 3) $\rightarrow 6$; (0, 4) $\rightarrow 16$. **Min $Z = -12$ at (4, 0).**

3. Maximise $Z = 5x + 3y$ subject to $3x + 5y \leq 15$, $5x + 2y \leq 10$, $x, y \geq 0$.

Intersection: $15x + 25y = 75$ and $15x + 6y = 30 \Rightarrow 19y = 45 \Rightarrow y = 45/19$, $x = 20/19$.Corners: (0, 0) $\rightarrow 0$; (2, 0) $\rightarrow 10$; $(20/19, 45/19) \rightarrow 100/19 + 135/19 = 235/19$; (0, 3) $\rightarrow 9$.

Max $Z = 235/19$ at $(20/19, 45/19)$.

4. Minimise $Z = 200x + 500y$ subject to $x + 2y \geq 10$, $3x + 4y \leq 24$, $x, y \geq 0$.

Intersection of $x + 2y = 10$ and $3x + 4y = 24$: $x = 4$, $y = 3$. Corners: $(0, 5) \rightarrow 2500$; $(4, 3) \rightarrow 2300$; $(0, 6) \rightarrow 3000$.

Min $Z = 2300$ at $(4, 3)$.

C. Long answer questions**1. A furniture dealer deals in tables and chairs. He has Rs 50,000 to invest and space for at most 60 pieces. A table costs Rs 2500 and a chair Rs 500. Profit is Rs 250 per table and Rs 75 per chair. How many of each should he buy for maximum profit?**

Let x tables, y chairs. Max $Z = 250x + 75y$.

Constraints: $x + y \leq 60$; $2500x + 500y \leq 50000 \Rightarrow 5x + y \leq 100$; $x, y \geq 0$.

Intersection: subtract $x + y = 60$ from $5x + y = 100$: $4x = 40 \Rightarrow x = 10$, $y = 50$.

Corner	Z
$(0, 0)$	0
$(20, 0)$	5000
$(10, 50)$	$2500 + 3750 = 6250$
$(0, 60)$	4500

10 tables and 50 chairs; maximum profit Rs 6250.

2. Reshma wishes to mix two foods P and Q so that the mixture has at least 8 units of vitamin A and 11 units of vitamin B. Food P costs Rs 60/kg and Q Rs 80/kg. One kg of P has 3 units of A and 5 units of B; one kg of Q has 4 units of A and 2 units of B. Find the minimum cost.

Let x kg of P and y kg of Q. Min $Z = 60x + 80y$.

Constraints: $3x + 4y \geq 8$; $5x + 2y \geq 11$; $x, y \geq 0$. The region is unbounded.

Intersection: $3x + 4y = 8$ and $10x + 4y = 22 \Rightarrow 7x = 14 \Rightarrow x = 2$, $y = 1/2$.

Corners: $(8/3, 0) \rightarrow 160$; $(2, 1/2) \rightarrow 120 + 40 = 160$; $(0, 11/2) \rightarrow 440$.

Smallest value 160. Check: the half-plane $60x + 80y < 160$, i.e. $3x + 4y < 8$, has no point in common with the region (which needs $3x + 4y \geq 8$). So the minimum is valid.

Minimum cost Rs 160, at every point on the segment joining $(8/3, 0)$ and $(2, 1/2)$.

3. Minimise and maximise $Z = x + 2y$ subject to $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$, $x, y \geq 0$.

Lines: $x + 2y = 100$, $y = 2x$, $2x + y = 200$.

Corners: $x + 2y = 100$ with $y = 2x \Rightarrow (20, 40)$; $y = 2x$ with $2x + y = 200 \Rightarrow (50, 100)$; $2x + y = 200$ with $x = 0 \Rightarrow (0, 200)$; $x + 2y = 100$ with $x = 0 \Rightarrow (0, 50)$.

Z: $(0, 50) \rightarrow 100$; $(20, 40) \rightarrow 100$; $(50, 100) \rightarrow 250$; $(0, 200) \rightarrow 400$.

Max $Z = 400$ at $(0, 200)$; min $Z = 100$ at all points on the segment joining $(0, 50)$ and $(20, 40)$.

4. A factory makes tennis rackets and cricket bats. A racket needs 1.5 hours of machine time and 3 hours of craftsman's time; a bat needs 3 hours of machine time and 1 hour of craftsman's time. In a day there are at most 42 machine hours and 24 craftsman hours. Profit is Rs 20 per racket and Rs 10 per bat. Find the daily production for maximum profit.

Let x rackets, y bats. $\text{Max } Z = 20x + 10y$.

Constraints: $1.5x + 3y \leq 42 \Rightarrow x + 2y \leq 28$; $3x + y \leq 24$; $x, y \geq 0$.

Intersection: $x = 28 - 2y \Rightarrow 84 - 6y + y = 24 \Rightarrow y = 12, x = 4$.

Corners: $(0, 0) \rightarrow 0$; $(8, 0) \rightarrow 160$; $(4, 12) \rightarrow 80 + 120 = 200$; $(0, 14) \rightarrow 140$.

4 rackets and 12 bats per day; maximum profit Rs 200.

QB-13. Probability

Key points

- Conditional probability: $P(A|B) = P(A \cap B)/P(B)$, $P(B) \neq 0$.
- Multiplication rule: $P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$.
- Independent events: $P(A \cap B) = P(A)P(B)$. Then A' and B' , A and B' are also independent.
- Total probability: if E_1, E_2 up to E_n partition the sample space, $P(A) = \sum P(E_i)P(A|E_i)$.
- **Bayes' theorem:** $P(E_i|A) = P(E_i)P(A|E_i)/\sum P(E_j)P(A|E_j)$.
- Random variable X with values x_i and probabilities p_i : $\sum p_i = 1$; mean $\mu = \sum x_i p_i$; variance $= \sum x_i^2 p_i - \mu^2$.
- **Binomial distribution** (n independent Bernoulli trials, success probability p , $q = 1 - p$): $P(X = r) = {}^n C_r p^r q^{n-r}$. Mean np , variance npq .

Mutually exclusive and independent are different: mutually exclusive events (with non-zero probabilities) can never be independent.

A. Multiple choice questions

1. If A and B are independent with $P(A) = 0.4$, $P(B) = 0.5$, then $P(A \cup B) =$ (a) 0.9 (b) 0.7 (c) 0.2 (d) 0.6
(b) $0.4 + 0.5 - 0.2$.
2. Mean of a binomial distribution is (a) npq (b) np (c) $\sqrt{npq}$ (d) nq
(b).
3. Variance of a binomial distribution is (a) np (b) npq (c) np^2 (d) n^2pq
(b).
4. Two coins are tossed. $P(\text{both heads} | \text{at least one head}) =$ (a) $1/4$ (b) $1/2$ (c) $1/3$ (d) $3/4$
(c) $(1/4)/(3/4)$.
5. In a probability distribution, $\sum p_i =$ (a) 0 (b) 1 (c) n (d) mean
(b).
6. A die is thrown. $P(\text{number greater than 4}) =$ (a) $1/6$ (b) $1/3$ (c) $1/2$ (d) $2/3$

(b) {5, 6}.

7. If $P(A) = 1/2$, $P(B) = 1/3$, $P(A \cap B) = 1/6$, then $P(A|B) =$ (a) $1/3$ (b) $1/2$ (c) $1/6$ (d) $2/3$
(b) $(1/6)/(1/3)$.

8. For $X \sim B(4, 1/2)$, $P(X = 2) =$ (a) $1/4$ (b) $3/8$ (c) $1/2$ (d) $1/16$
(b) $6/16$.

9. If $P(A) = 0.3$, $P(B) = 0.6$ and A, B independent, $P(\text{neither A nor B}) =$ (a) 0.28 (b) 0.18
(c) 0.1 (d) 0.72
(a) 0.7×0.4 .

10. If A and B are mutually exclusive, $P(A \cap B) =$ (a) $P(A)P(B)$ (b) 0 (c) 1 (d) $P(A) + P(B)$
(b).

B. Short answer questions

1. $P(A) = 0.8$, $P(B) = 0.5$, $P(B|A) = 0.4$. Find $P(A \cap B)$, $P(A|B)$ and $P(A \cup B)$.

$P(A \cap B) = 0.8 \times 0.4 = 0.32$. $P(A|B) = 0.32/0.5 = 0.64$. $P(A \cup B) = 0.8 + 0.5 - 0.32 = 0.98$. Answers **0.32, 0.64, 0.98**.

2. A family has two children. Find $P(\text{both are boys} \mid \text{the elder is a boy})$.

$S = \{BB, BG, GB, GG\}$ (elder first). Elder boy: $\{BB, BG\}$. Both boys: $\{BB\}$. Probability = **$1/2$** .

3. A die is thrown twice and the sum is 8. Find the probability that 4 appeared at least once.

Sum 8: (2,6), (3,5), (4,4), (5,3), (6,2) — 5 outcomes. With a 4: (4,4) only. Probability = **$1/5$** .

4. Find the probability distribution and mean of the number of heads in two tosses of a coin.

$X = 0, 1, 2$ with $P = 1/4, 1/2, 1/4$. Mean = $0 + 1/2 + 2/4 =$ **1**.

5. $P(X = x) = kx$ for $x = 1, 2, 3, 4$. Find k and the mean.

$k(1 + 2 + 3 + 4) = 1 \Rightarrow k =$ **$1/10$** . Mean = $\sum x \cdot kx = (1 + 4 + 9 + 16)/10 =$ **3**.

6. A coin is tossed 6 times. Find $P(\text{exactly 4 heads})$.

${}^6C_4(1/2)^6 = 15/64$. Answer **$15/64$** .

7. Find the mean and variance of a binomial distribution with $n = 10$, $p = 0.4$.

Mean = 4; variance = $10 \times 0.4 \times 0.6 =$ **2.4**.

8. Two cards are drawn without replacement from a pack of 52. Find $P(\text{both are kings})$.

$(4/52)(3/51) = 12/2652 =$ **$1/221$** .

C. Long answer questions

1. Bag I has 3 red and 4 black balls; Bag II has 5 red and 6 black balls. A bag is chosen at random and a ball drawn from it is red. Find the probability that it came from Bag II.

E_1 : Bag I, E_2 : Bag II; $P(E_1) = P(E_2) = 1/2$. A: red ball. $P(A|E_1) = 3/7$, $P(A|E_2) = 5/11$.
 $P(E_2|A) = (1/2)(5/11)/[(1/2)(3/7) + (1/2)(5/11)] = (5/11)/(3/7 + 5/11) = (5/11)/(68/77) = 35/68$.

35/68.

2. Machines A, B, C produce 25%, 35%, 40% of the bolts in a factory, of which 5%, 4% and 2% respectively are defective. A bolt chosen at random is defective. Find the probability that it was made by machine B.

$P(D) = 0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02 = 0.0125 + 0.014 + 0.008 = 0.0345$.
 $P(B|D) = 0.014/0.0345 = 140/345 = 28/69$.

3. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probabilities of an accident are 0.01, 0.03 and 0.15 respectively. One insured person meets with an accident. What is the probability that he is a scooter driver?

Total 12000. $P(S) = 1/6$, $P(C) = 1/3$, $P(T) = 1/2$.
 $P(A) = (1/6)(0.01) + (1/3)(0.03) + (1/2)(0.15)$. Multiply all by 12000 to use counts:
 expected accidents = $20 + 120 + 900 = 1040$.
 $P(S|A) = 20/1040 = 1/52$.

4. Two cards are drawn one by one with replacement from a well-shuffled pack of 52. Find the probability distribution, mean and variance of the number of aces.

$p = P(\text{ace}) = 1/13$, $q = 12/13$. $X = 0, 1, 2$.
 $P(0) = (12/13)^2 = 144/169$; $P(1) = 2(1/13)(12/13) = 24/169$; $P(2) = 1/169$.
 Mean = $0 + 24/169 + 2/169 = 26/169 = 2/13$.
 $E(X^2) = 24/169 + 4/169 = 28/169$. Variance = $28/169 - 4/169 = 24/169$. (Check: $npq = 2 \times (1/13) \times (12/13) = 24/169$ ✓.)

5. A die is thrown 6 times. Getting an odd number is a success. Find $P(5 \text{ successes})$, $P(\text{at least } 5 \text{ successes})$ and $P(\text{at most } 5 \text{ successes})$.

$n = 6$, $p = q = 1/2$.
 $P(5) = {}^6C_5(1/2)^6 = 6/64 = 3/32$.
 $P(X \geq 5) = (6 + 1)/64 = 7/64$.
 $P(X \leq 5) = 1 - P(6) = 1 - 1/64 = 63/64$.

6. 10% of the bulbs produced by a factory are defective. Five bulbs are chosen at random. Find P(none defective) and P(at most one defective).

$$n = 5, p = 0.1, q = 0.9.$$

$$P(0) = 0.9^5 = \mathbf{0.59049}.$$

$$P(1) = 5 \times 0.1 \times 0.9^4 = 0.5 \times 0.6561 = 0.32805.$$

$$P(X \leq 1) = 0.59049 + 0.32805 = \mathbf{0.91854}.$$

Solved Sample Paper 1

Full marks: 100 | Time: 3 hours. Section A: Q1 (10 MCQs $\times$ 1) and Q2 (10 very short answers $\times$ 1) = 20 marks. Section B: Q3 (8 short answers $\times$ 3) = 24 marks. Section C: Q4 to Q13, answer any eight (8×7) = 56 marks. This model follows the usual CHSE layout of objective, short and long questions; always check the latest official blueprint before the exam.

HOW TO USE THESE PAPERS

Sit with a clock for 3 hours, attempt the paper on your own, and only then check the solutions. Mark yourself honestly and note which chapters cost you marks.

Section A

Q1. Choose the correct option.

(a) If $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 3x + 2$, then $f^{-1}(x) =$ (i) $(x + 2)/3$ (ii) $(x - 2)/3$ (iii) $3x - 2$ (iv) $1/(3x + 2)$

(ii). Put $y = 3x + 2 \Rightarrow x = (y - 2)/3$.

(b) The principal value of $\sin^{-1}(-1/2)$ is (i) $-\pi/6$ (ii) $7\pi/6$ (iii) $5\pi/6$ (iv) $\pi/6$

(i). Range of $\sin^{-1}$ is $[-\pi/2, \pi/2]$.

(c) If A is a 3×3 matrix with $|A| = 4$, then $|\text{adj } A| =$ (i) 4 (ii) 8 (iii) 16 (iv) 64

(iii). $|\text{adj } A| = |A|^{n-1} = 4^2$.

(d) If A is 2×3 and B is 3×4 , the order of AB is (i) 3×3 (ii) 2×4 (iii) 4×2 (iv) not defined (ii).

(e) $d/dx(\sin^{-1}x) =$ (i) $1/\sqrt{1-x^2}$ (ii) $-1/\sqrt{1-x^2}$ (iii) $1/(1+x^2)$ (iv) $\cos^{-1}x$

(i).

(f) The radius of a circle increases at 2 cm/s. The rate of increase of its area when $r = 5$ cm is (i) 10π (ii) 20π (iii) 25π (iv) $5\pi \text{ cm}^2/\text{s}$

(ii). $dA/dt = 2\pi r dr/dt = 2\pi(5)(2)$.

(g) $\int_0^{\pi/2} \cos^2 x dx =$ (i) $\pi/2$ (ii) $\pi/4$ (iii) 1 (iv) 0

(ii).

(h) The degree of $(dy/dx)^2 + y = 0$ is (i) 1 (ii) 2 (iii) 0 (iv) not defined

(ii).

(i) $|\hat{i} \times \hat{j}| =$ (i) 0 (ii) 1 (iii) $\sqrt{2}$ (iv) -1

(ii). $\hat{i} \times \hat{j} = \hat{k}$.

(j) A and B are independent, $P(A) = 0.6$, $P(B) = 0.3$. Then $P(A \cap B) =$ (i) 0.9 (ii) 0.18 (iii) 0.3 (iv) 0.5

(ii).

Q2. Answer in one word or one sentence.

(a) How many functions are there from a set with 2 elements to a set with 3 elements?

$$3^2 = 9.$$

(b) Find $\tan^{-1}(1) + \cos^{-1}(-1/2)$.

$$\pi/4 + 2\pi/3 = 11\pi/12.$$

(c) For what x is the determinant $|x \ 2; \ 3 \ x|$ equal to 0?

$$x^2 - 6 = 0 \Rightarrow x = \pm\sqrt{6}.$$

(d) For what k is $[[k, 2], [3, 6]]$ singular?

$$6k - 6 = 0 \Rightarrow k = 1.$$

(e) Differentiate $e^{\sin x}$.

$$\cos x \cdot e^{\sin x}.$$

(f) Find the slope of the tangent to $y = x^3$ at $x = 1$.

$$3x^2 = 3.$$

(g) Evaluate $\int e^{2x} dx$.

$$e^{2x}/2 + C.$$

(h) Write the order of $d^3y/dx^3 + y = 0$.

$$3.$$

(i) Write the direction cosines of a line equally inclined to the three axes.

$$\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3} \text{ (all of the same sign).}$$

(j) Find the mean of a binomial distribution with $n = 20$, $p = 1/4$.

$$np = 5.$$

Section B

Q3. Answer the following (3 marks each).

(a) Show that $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = 2x + 1$ is a bijection.

$$\text{One-one: } f(x_1) = f(x_2) \Rightarrow 2x_1 + 1 = 2x_2 + 1 \Rightarrow x_1 = x_2.$$

Onto: for any $y \in \mathbb{R}$, $x = (y - 1)/2 \in \mathbb{R}$ and $f(x) = y$. Hence f is one-one and onto, i.e. a bijection.

(b) Prove that $\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4$.

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}[(x + y)/(1 - xy)] \quad (xy < 1). \quad (1/2 + 1/3)/(1 - 1/6) = (5/6)/(5/6) = 1. \quad \tan^{-1}1 = \pi/4. \text{ Proved.}$$

(c) If $A = [[1, 2], [3, 4]]$, find $A^2 - 5A$.

$$A^2 = [[1 + 6, 2 + 8], [3 + 12, 6 + 16]] = [[7, 10], [15, 22]]. \quad 5A = [[5, 10], [15, 20]]. \quad A^2 - 5A = \mathbf{[[2, 0], [0, 2]] = 2I}.$$

(d) Differentiate $x^{\sin x}$ ($x > 0$).

$$\log y = \sin x \log x. \quad (1/y)y' = \cos x \log x + (\sin x)/x. \quad \mathbf{y' = x^{\sin x}[\cos x \log x + (\sin x)/x]}.$$

(e) Find the intervals in which $f(x) = x^2 - 4x + 6$ is increasing or decreasing.

$f'(x) = 2x - 4$. $f' > 0$ for $x > 2$ and $f' < 0$ for $x < 2$. **Increasing on $(2, \infty)$, decreasing on $(-\infty, 2)$.**

(f) Evaluate $\int x \sin x \, dx$.

$u = x$, $v = \sin x$: $x(-\cos x) - \int 1 \cdot (-\cos x) \, dx = -x \cos x + \sin x + C$.

(g) Solve $dy/dx = e^x - y$.

$e^y \, dy = e^x \, dx \Rightarrow e^y = e^x + C$.

(h) If $\hat{a}$ is a unit vector and $(x - \hat{a}) \cdot (x + \hat{a}) = 12$, find $|x|$.

$|x|^2 - |\hat{a}|^2 = 12 \Rightarrow |x|^2 = 13 \Rightarrow |x| = \sqrt{13}$.

Section C (answer any eight)

Q4. Solve by the matrix method: $x + y + z = 6$, $x - y + z = 2$, $2x + y - z = 1$.

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, $X = [x, y, z]^T$, $B = [6, 2, 1]^T$.

$|A| = 1(1 - 1) - 1(-1 - 2) + 1(1 + 2) = 0 + 3 + 3 = 6 \neq 0$, so A^{-1} exists.

Cofactors: $C_{11} = 0$, $C_{12} = 3$, $C_{13} = 3$; $C_{21} = 2$, $C_{22} = -3$, $C_{23} = 1$; $C_{31} = 2$, $C_{32} = 0$, $C_{33} = -2$.

$\text{adj } A = (\text{transpose}) \begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$. $A^{-1} = (1/6) \text{adj } A$.

$X = A^{-1}B = (1/6)[0 + 4 + 2, 18 - 6 + 0, 18 + 2 - 2]^T = (1/6)[6, 12, 18]^T = [1, 2, 3]^T$.

$x = 1$, $y = 2$, $z = 3$.

Q5. Prove that $|1 \ a \ a^2; 1 \ b \ b^2; 1 \ c \ c^2| = (a - b)(b - c)(c - a)$.

$R_1 \rightarrow R_1 - R_2$, $R_2 \rightarrow R_2 - R_3$:

$\Delta = |0 \ a - b \ a^2 - b^2; 0 \ b - c \ b^2 - c^2; 1 \ c \ c^2|$.

Take $(a - b)$ from R_1 and $(b - c)$ from R_2 : $\Delta = (a - b)(b - c)|0 \ 1 \ a + b; 0 \ 1 \ b + c; 1 \ c \ c^2|$.

Expand along C_1 : $\Delta = (a - b)(b - c) \cdot 1 \cdot [(b + c) - (a + b)] \cdot (\text{sign})$ — explicitly, the

cofactor of the entry 1 in position (3, 1) is $+|1 \ a + b; 1 \ b + c| = (b + c) - (a + b) = c - a$.

Hence $\Delta = (a - b)(b - c)(c - a)$.

Q6. If $y = (\sin^{-1}x)^2$, prove that $(1 - x^2)y_2 - xy_1 - 2 = 0$.

$y_1 = 2 \sin^{-1}x / \sqrt{1 - x^2} \Rightarrow \sqrt{1 - x^2} y_1 = 2 \sin^{-1}x$.

Square: $(1 - x^2)y_1^2 = 4(\sin^{-1}x)^2 = 4y$.

Differentiate: $(1 - x^2) \cdot 2y_1 y_2 - 2x y_1^2 = 4y_1$.

Divide by $2y_1$: $(1 - x^2)y_2 - xy_1 = 2$, i.e. **$(1 - x^2)y_2 - xy_1 - 2 = 0$.**

Q7. Show that a right circular cylinder of given volume, open at the top, has the least total surface area when its height equals its radius.

$V = \pi r^2 h$ (fixed) $\Rightarrow h = V/(\pi r^2)$. Surface (base + curved) $S = \pi r^2 + 2\pi r h = \pi r^2 + 2V/r$.

$dS/dr = 2\pi r - 2V/r^2 = 0 \Rightarrow \pi r^3 = V = \pi r^2 h \Rightarrow h = r$.

$d^2S/dr^2 = 2\pi + 4V/r^3 > 0 \Rightarrow$ minimum. **Height = radius.**

Q8. Evaluate $\int_0^{\pi/4} \log(1 + \tan x) dx$.

$I = \int_0^{\pi/4} \log(1 + \tan x) dx$. By P4, replace x by $\pi/4 - x$: $\tan(\pi/4 - x) = (1 - \tan x)/(1 + \tan x)$.

$1 + \tan(\pi/4 - x) = 2/(1 + \tan x)$. So $I = \int_0^{\pi/4} [\log 2 - \log(1 + \tan x)] dx = (\pi/4) \log 2 - I$.

$2I = (\pi/4) \log 2 \Rightarrow \mathbf{I = (\pi/8) \log 2}$.

Q9. Find the area of the smaller region bounded by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$.

The line cuts the circle at (2, 0) and (0, 2). The smaller region is in the first quadrant between the arc and the chord.

Area $= \int_0^2 [\sqrt{4 - x^2} - (2 - x)] dx = (\text{quarter circle}) - (\text{triangle}) = \pi - (1/2)(2)(2) = \mathbf{\pi - 2 \text{ sq units}}$.

Q10. Find the shortest distance between $\mathbf{r = 4\hat{i} - \hat{j} + \lambda(\hat{i} + 2\hat{j} - 3\hat{k})}$ and $\mathbf{r = \hat{i} - \hat{j} + 2\hat{k} + \mu(2\hat{i} + 4\hat{j} - 5\hat{k})}$.

$\mathbf{b_1} \times \mathbf{b_2} = (1, 2, -3) \times (2, 4, -5) = (2(-5) - (-3)4, -(1(-5) - (-3)2), 1 \cdot 4 - 2 \cdot 2) = (2, -1, 0)$; magnitude $\sqrt{5}$.

$\mathbf{a_2} - \mathbf{a_1} = (1 - 4, -1 + 1, 2 - 0) = (-3, 0, 2)$. Dot $= -6$.

$d = 6/\sqrt{5} = \mathbf{6\sqrt{5}/5 \text{ units}}$.

Q11. Maximise $Z = 4x + y$ subject to $x + y \leq 50$, $3x + y \leq 90$, $x, y \geq 0$.

Intersection: subtract $\Rightarrow 2x = 40 \Rightarrow x = 20, y = 30$.

Corners: (0, 0) $\rightarrow 0$; (30, 0) $\rightarrow 120$; (20, 30) $\rightarrow 110$; (0, 50) $\rightarrow 50$.

Max $Z = 120$ at (30, 0).

Q12. Solve $dy/dx - y/x = 2x^2$.

$P = -1/x \Rightarrow \text{IF} = e^{-\log x} = 1/x$.

$y/x = \int 2x^2 \cdot (1/x) dx = x^2 + C \Rightarrow \mathbf{y = x^3 + Cx}$.

Q13. A man speaks the truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

E_1 : six occurs, $P = 1/6$; E_2 : six does not occur, $P = 5/6$. A: he reports six.

$P(A|E_1) = 3/4$ (truth), $P(A|E_2) = 1/4$ (lie).

$P(E_1|A) = (1/6)(3/4)/[(1/6)(3/4) + (5/6)(1/4)] = (3/24)/(8/24) = \mathbf{3/8}$.

Solved Sample Paper 2

Full marks: 100 | Time: 3 hours. Same pattern as Paper 1.

Section A

Q1. Choose the correct option.

(a) The relation $R = \{(1, 1), (2, 2), (3, 3)\}$ on $\{1, 2, 3\}$ is (i) reflexive only (ii) symmetric only (iii) an equivalence relation (iv) not transitive
(iii).

(b) $\cos^{-1}(\cos 7\pi/6) =$ (i) $7\pi/6$ (ii) $5\pi/6$ (iii) $\pi/6$ (iv) $-\pi/6$

(ii). $7\pi/6$ is outside $[0, \pi]$; $\cos 7\pi/6 = \cos 5\pi/6$.

(c) If A is a skew-symmetric matrix of order 3, $|A| =$ (i) 1 (ii) -1 (iii) 0 (iv) 3
(iii).

(d) If $A^2 = A$, then $(I + A)^2 - 3A =$ (i) I (ii) A (iii) $2A$ (iv) 0

(i). $I + 2A + A^2 - 3A = I + 3A - 3A = I$.

(e) $d/dx(\log|x|) =$ (i) $1/|x|$ (ii) $1/x$ (iii) x (iv) $\log x$
(ii).

(f) $f(x) = x^3$ at $x = 0$ has (i) a maximum (ii) a minimum (iii) neither (iv) a discontinuity
(iii). f' does not change sign; it is a point of inflection.

(g) $\int (\log x)/x \, dx =$ (i) $\log x + C$ (ii) $(\log x)^2/2 + C$ (iii) $1/x + C$ (iv) $x \log x + C$
(ii).

(h) The integrating factor of $dy/dx + y = x$ is (i) e^x (ii) e^{-x} (iii) x (iv) $\log x$
(i).

(i) The angle between $\hat{i}$ and $\hat{i} + \hat{j}$ is (i) 30° (ii) 45° (iii) 60° (iv) 90°
(ii). $\cos \theta = 1/\sqrt{2}$.

(j) If $P(B) = 0.4$ and $P(A \cap B) = 0.2$, then $P(A|B) =$ (i) 0.08 (ii) 0.2 (iii) 0.5 (iv) 0.6
(iii).

Q2. Answer in one word or one sentence.

(a) If $f(x) = x^2$ and $g(x) = x + 1$, find $(f \circ g)(2)$.
 $f(3) = 9$.

(b) Find $\sin(\sin^{-1}(1/3) + \cos^{-1}(1/3))$.
 $\sin(\pi/2) = 1$.

(c) A is 2×2 and $|A| = 3$. Find $|2A|$.
 $2^2 \times 3 = 12$.

(d) How many 2×2 matrices have every entry 0 or 1?
 $2^4 = 16$.

(e) Differentiate $\tan^{-1}(x^2)$.
 $2x/(1 + x^4)$.

(f) Find the maximum value of $\sin x + \cos x$.

$\sqrt{2}$.

(g) Evaluate $\int_{-\pi/2}^{\pi/2} \sin x \, dx$.

0 (odd function).

(h) Solve $dy/dx = 2x$.

$y = x^2 + C$.

(i) Find $\hat{i} \cdot \hat{i} + \hat{j} \cdot \hat{k}$.

$1 + 0 = 1$.

(j) Three coins are tossed. Find P(at least one head).

$1 - 1/8 = 7/8$.

Section B

Q3. Answer the following (3 marks each).

(a) Show that $R = \{(a, b): a - b \text{ is divisible by } 5\}$ on Z is an equivalence relation.

Reflexive: $a - a = 0$ is divisible by 5. Symmetric: if $5 \mid (a - b)$, then $5 \mid (b - a) = -(a - b)$. Transitive: if $5 \mid (a - b)$ and $5 \mid (b - c)$, then $5 \mid (a - b) + (b - c) = a - c$. Hence R is an equivalence relation.

(b) Solve $\tan^{-1} 2x + \tan^{-1} 3x = \pi/4$.

$\tan^{-1}[5x/(1 - 6x^2)] = \pi/4 \Rightarrow 5x = 1 - 6x^2 \Rightarrow 6x^2 + 5x - 1 = 0 \Rightarrow (6x - 1)(x + 1) = 0$.
 $x = -1$ makes the left side negative, so it is rejected. **$x = 1/6$.**

(c) Find the inverse of $A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

$|A| = 4 - 3 = 1$. $\text{adj } A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$. **$A^{-1} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.**

(d) If $x = a \cos \theta$, $y = a \sin \theta$, find dy/dx at $\theta = \pi/4$.

$dy/dx = (a \cos \theta)/(-a \sin \theta) = -\cot \theta$. At $\pi/4$: **-1 .**

(e) Find the equation of the tangent to $y = x^2 - 2x + 7$ at $(2, 7)$.

$dy/dx = 2x - 2 = 2$ at $x = 2$. $y - 7 = 2(x - 2) \Rightarrow$ **$y = 2x + 3$.**

(f) Evaluate $\int dx/(9 - x^2)$.

$(1/6) \log|(3 + x)/(3 - x)| + C$.

(g) Find the area bounded by $y = 2x$, the x -axis and $x = 3$.

$\int_0^3 2x \, dx = 9$. **9 sq units.**

(h) $P(A) = 1/4$, $P(B) = 1/3$, $P(A|B) = 1/2$. Find $P(A \cup B)$.

$P(A \cap B) = (1/2)(1/3) = 1/6$. $P(A \cup B) = 1/4 + 1/3 - 1/6 =$ **$5/12$.**

Section C (answer any eight)**Q4. Solve by the matrix method: $x - y + 2z = 1$, $2y - 3z = 1$, $3x - 2y + 4z = 2$.**
 $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$. $|A| = 1(8 - 6) + 1(0 + 9) + 2(0 - 6) = 2 + 9 - 12 = -1$.

 Cofactors: $C_{11} = 2$, $C_{12} = -9$, $C_{13} = -6$; $C_{21} = 0$, $C_{22} = -2$, $C_{23} = -1$; $C_{31} = -1$, $C_{32} = 3$, $C_{33} = 2$.

 $\text{adj } A = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$; $A^{-1} = -\text{adj } A = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$.

 $X = A^{-1}[1, 1, 2]^T = [-2 + 0 + 2, 9 + 2 - 6, 6 + 1 - 4]^T = [0, 5, 3]^T$.

 $x = 0, y = 5, z = 3$. (Check: $0 - 5 + 6 = 1$ ✓.)
Q5. Prove that $|b + c \ a \ a; b \ c \ + \ a \ b; c \ c \ a \ + \ b| = 4abc$.
 $R_1 \rightarrow R_1 - R_2 - R_3$: $R_1 = (0, -2c, -2b)$.

 $\Delta = |0 \ -2c \ -2b; b \ c \ + \ a \ b; c \ c \ a \ + \ b|$. Expand along R_1 :

$$\Delta = 0 + 2c[b(a + b) - bc] - 2b[bc - c(c + a)] = 2c(ab + b^2 - bc) - 2b(bc - c^2 - ac)$$

$$= 2abc + 2b^2c - 2bc^2 - 2b^2c + 2bc^2 + 2abc = 4abc.$$
Q6. If $y = Ae^{mx} + Be^{nx}$, show that $y'' - (m + n)y' + mny = 0$.
 $y' = mAe^{mx} + nBe^{nx}$; $y'' = m^2Ae^{mx} + n^2Be^{nx}$.

 $y'' - (m + n)y' + mny = Ae^{mx}[m^2 - (m + n)m + mn] + Be^{nx}[n^2 - (m + n)n + mn] = Ae^{mx}(0) + Be^{nx}(0) = 0$. Proved.
Q7. Find the maximum area of a rectangle inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$.
 Take a vertex at $(a \cos \theta, b \sin \theta)$. Sides $2a \cos \theta$ and $2b \sin \theta$. Area $A = 4ab \sin \theta \cos \theta = 2ab \sin 2\theta$.

 $dA/d\theta = 4ab \cos 2\theta = 0 \Rightarrow \theta = \pi/4$; $d^2A/d\theta^2 = -8ab \sin 2\theta < 0 \Rightarrow$ maximum.

Maximum area = $2ab$.
Q8. Evaluate $\int dx/[(1 + x)(1 + x^2)]$.
 $1/[(1 + x)(1 + x^2)] = A/(1 + x) + (Bx + C)/(1 + x^2) \Rightarrow 1 = A(1 + x^2) + (Bx + C)(1 + x)$.

 $x = -1$: $1 = 2A \Rightarrow A = 1/2$. Coefficient of x^2 : $A + B = 0 \Rightarrow B = -1/2$. Constant: $A + C = 1 \Rightarrow C = 1/2$.

$$f = (1/2) \log|1 + x| - (1/2) \int x/(1 + x^2) dx + (1/2) \int dx/(1 + x^2)$$

$$= (1/2) \log|1 + x| - (1/4) \log(1 + x^2) + (1/2) \tan^{-1}x + C.$$

Q9. Find the area of the region bounded by the parabola $x^2 = 4y$ and the line $x = 4y - 2$.

Line: $y = (x + 2)/4$. Intersection: $x^2 = x + 2 \Rightarrow x = -1, 2$.

$$\text{Area} = \int_{-1}^2 [(x + 2)/4 - x^2/4] dx = (1/4)[x^2/2 + 2x - x^3/3]_{-1}^2 = (1/4)[(2 + 4 - 8/3) - (1/2 - 2 + 1/3)] = (1/4)[10/3 + 7/6] = (1/4)(27/6) = \mathbf{9/8 \text{ sq units.}}$$

Q10. Find the equation of the plane through $(-1, 3, 2)$ perpendicular to the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$.

Normal = $(1, 2, 3) \times (3, 3, 1) = (2 - 9, -(1 - 9), 3 - 6) = (-7, 8, -3)$.

$$-7(x + 1) + 8(y - 3) - 3(z - 2) = 0 \Rightarrow -7x + 8y - 3z - 25 = 0 \Rightarrow \mathbf{7x - 8y + 3z + 25 = 0.}$$

Q11. Minimise $Z = 3x + 5y$ subject to $x + 3y \geq 3$, $x + y \geq 2$, $x, y \geq 0$.

Intersection: subtract $\Rightarrow 2y = 1 \Rightarrow y = 1/2$, $x = 3/2$. Unbounded region with corners $(3, 0) \rightarrow 9$, $(3/2, 1/2) \rightarrow 7$, $(0, 2) \rightarrow 10$.

Check: $3x + 5y = 2(x + y) + (x + 3y) \geq 4 + 3 = 7$ everywhere in the region, so 7 is truly the minimum.

Min $Z = 7$ at $(3/2, 1/2)$.

Q12. Solve $x \, dy/dx = y - x \tan(y/x)$.

$dy/dx = y/x - \tan(y/x)$. Put $y = vx$: $v + x \, dv/dx = v - \tan v \Rightarrow x \, dv/dx = -\tan v$.

$$\cot v \, dv = -dx/x \Rightarrow \log|\sin v| = -\log|x| + \log C \Rightarrow x \sin v = C.$$

$x \sin(y/x) = C$.

Q13. A lot of 10 items has 3 defectives. Two items are drawn without replacement. Find the probability distribution, mean and variance of the number of defectives.

$${}^{10}C_2 = 45. P(0) = {}^7C_2/45 = 21/45 = 7/15; P(1) = (3 \times 7)/45 = 7/15; P(2) = {}^3C_2/45 = 1/15.$$

$$\text{Mean} = 7/15 + 2/15 = 9/15 = \mathbf{3/5}.$$

$$E(X^2) = 7/15 + 4/15 = 11/15. \text{ Variance} = 11/15 - 9/25 = 55/75 - 27/75 = \mathbf{28/75}.$$

Solved Sample Paper 3

Full marks: 100 | Time: 3 hours. Section A: Q1 (10 MCQs $\times$ 1) and Q2 (10 very short answers $\times$ 1) = 20 marks. Section B: Q3 (8 short answers $\times$ 3) = 24 marks. Section C: Q4 to Q13, answer any eight (8×7) = 56 marks. This is a practice model in the usual CHSE layout; always check the latest official blueprint before the exam.

TIME PLAN

Section A: 25 minutes. Section B: 45 minutes. Section C: 95 minutes (about 12 minutes per long question). Keep 15 minutes at the end to recheck signs, limits and final answers.

Section A

Q1. Choose the correct option.

(a) The relation $R = \{(1, 1), (2, 2), (3, 3), (1, 2)\}$ on $\{1, 2, 3\}$ is (i) reflexive and symmetric (ii) reflexive and transitive but not symmetric (iii) an equivalence relation (iv) symmetric only

(ii). $(1, 2) \in R$ but $(2, 1) \notin R$, so it is not symmetric; every pair needed for transitivity is present.

(b) $\cos^{-1}(\cos 7\pi/6) =$ (i) $7\pi/6$ (ii) $5\pi/6$ (iii) $\pi/6$ (iv) $-\pi/6$

(ii). $7\pi/6$ is outside $[0, \pi]$; $\cos 7\pi/6 = \cos 5\pi/6$ and $5\pi/6 \in [0, \pi]$.

(c) The matrix $\begin{bmatrix} 2 & 3 \\ 4 & k \end{bmatrix}$ is singular when $k =$ (i) 3 (ii) 4 (iii) 6 (iv) 8

(iii). $2k - 12 = 0 \Rightarrow k = 6$.

(d) If A is a skew-symmetric matrix of order 3, then $|A| =$ (i) 1 (ii) -1 (iii) 0 (iv) $|A|^2$

(iii). $|A| = |A^T| = |-A| = (-1)^3|A| \Rightarrow |A| = 0$.

(e) $d/dx(e^{\sin x}) =$ (i) $e^{\sin x}$ (ii) $e^{\cos x}$ (iii) $\cos x \cdot e^{\sin x}$ (iv) $-\cos x \cdot e^{\sin x}$

(iii). Chain rule.

(f) $f(x) = x^3 - 3x$ has a local maximum at $x =$ (i) 1 (ii) -1 (iii) 0 (iv) 3

(ii). $f' = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$; $f'(-1) = -6 < 0$.

(g) $\int dx/(x \log x) =$ (i) $\log x + C$ (ii) $\log|\log x| + C$ (iii) $1/\log x + C$ (iv) $(\log x)^2/2 + C$

(ii). Put $t = \log x$.

(h) The order of $d^3y/dx^3 + (dy/dx)^2 + y = 0$ is (i) 1 (ii) 2 (iii) 3 (iv) 6

(iii).

(i) If $|a| = 2$, $|b| = \sqrt{3}$ and $a \cdot b = 3$, the angle between a and b is (i) $\pi/6$ (ii) $\pi/4$ (iii) $\pi/3$ (iv) $\pi/2$

(i). $\cos \theta = 3/(2\sqrt{3}) = \sqrt{3}/2$.

(j) If $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/6$, then $P(A | B) =$ (i) $1/3$ (ii) $1/2$ (iii) $1/6$ (iv) $2/3$

(ii). $(1/6)/(1/3) = 1/2$.

Q2. Answer in one word or one sentence.

(a) How many binary operations can be defined on the set {a, b}?

$$2^{2 \times 2} = 2^4 = 16.$$

(b) Evaluate $\sin[\pi/3 - \sin^{-1}(-1/2)]$.

$$\sin(\pi/3 + \pi/6) = \sin(\pi/2) = 1.$$

(c) If $[[x + y, 2], [5, x - y]] = [[6, 2], [5, 2]]$, find x and y.

$$x + y = 6, x - y = 2 \Rightarrow x = 4, y = 2.$$

(d) Differentiate $\log(\sec x + \tan x)$.

$$(\sec x \tan x + \sec^2 x)/(\sec x + \tan x) = \sec x.$$

(e) Find the slope of the tangent to $y = x^2 - 3x$ at $x = 2$.

$$dy/dx = 2x - 3 = 1.$$

(f) Evaluate $\int_{-1}^1 x^3 \cos x \, dx$.

0, since the integrand is odd.

(g) Write the integrating factor of $dy/dx + y/x = x^2$.

$$e^{\int dx/x} = x \quad (x > 0).$$

(h) Find the projection of $\mathbf{a} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ on $\mathbf{b} = \hat{i} + 2\hat{j} + \hat{k}$.

$$\mathbf{a} \cdot \mathbf{b}/|\mathbf{b}| = (2 + 6 + 2)/\sqrt{6} = 10/\sqrt{6}.$$

(i) Write the direction cosines of a line equally inclined to the three axes.

$$l = m = n \text{ and } 3l^2 = 1 \Rightarrow (\pm 1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3}), \text{ all signs alike.}$$

(j) A and B are mutually exclusive with $P(A) = 0.4$, $P(B) = 0.5$. Find $P(A \cup B)$.

$$0.4 + 0.5 = 0.9.$$

Section B

Q3. Answer the following (3 marks each).

(a) Show that the relation R on Z given by $a R b \Leftrightarrow 3 \text{ divides } (a - b)$ is an equivalence relation.

Reflexive: $a - a = 0 = 3 \cdot 0$, so $a R a$.

Symmetric: if $a - b = 3k$, then $b - a = 3(-k)$, so $b R a$.

Transitive: if $a - b = 3k$ and $b - c = 3m$, then $a - c = 3(k + m)$, so $a R c$.

Hence R is an equivalence relation.

(b) Prove that $2 \tan^{-1}(1/3) + \tan^{-1}(1/7) = \pi/4$.

$$2 \tan^{-1}(1/3) = \tan^{-1}[(2/3)/(1 - 1/9)] = \tan^{-1}(3/4).$$

$$\tan^{-1}(3/4) + \tan^{-1}(1/7) = \tan^{-1}[(3/4 + 1/7)/(1 - 3/28)] = \tan^{-1}[(25/28)/(25/28)] = \tan^{-1}1 = \pi/4.$$

(c) Find the inverse of $A = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$.

$$|A| = 6 - 4 = 2 \neq 0. \text{ adj } A = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}.$$

$$A^{-1} = (1/2)\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 \\ -2 & 3/2 \end{bmatrix}. \text{ Check: } AA^{-1} = I.$$

(d) Differentiate $y = x^{\sin x}$, $x > 0$.

$\log y = \sin x \cdot \log x$. Differentiate: $(1/y) dy/dx = \cos x \cdot \log x + \sin x/x$.

$$dy/dx = x^{\sin x}(\cos x \log x + \sin x/x).$$

(e) Find the intervals in which $f(x) = 2x^3 - 9x^2 + 12x + 5$ is increasing or decreasing.

$$f'(x) = 6x^2 - 18x + 12 = 6(x - 1)(x - 2).$$

$f' > 0$ for $x < 1$ and $x > 2$; $f' < 0$ for $1 < x < 2$.

Increasing on $(-\infty, 1)$ and $(2, \infty)$; decreasing on $(1, 2)$.

(f) Evaluate $\int x \sin x \, dx$.

By parts with $u = x$, $dv = \sin x \, dx$: $\int x \sin x \, dx = -x \cos x + \int \cos x \, dx = -x \cos x + \sin x + C$.

(g) Find a unit vector perpendicular to both $a = \hat{i} + \hat{j} + \hat{k}$ and $b = \hat{i} + 2\hat{j} + 3\hat{k}$.

$$a \times b = \hat{i}(3 - 2) - \hat{j}(3 - 1) + \hat{k}(2 - 1) = \hat{i} - 2\hat{j} + \hat{k}; |a \times b| = \sqrt{6}.$$

$$\pm(\hat{i} - 2\hat{j} + \hat{k})/\sqrt{6}.$$

(h) A die is thrown twice. Find the probability that the sum is 8, given that the first throw shows an even number.

B: first throw even, $n(B) = 18$. $A \cap B = \{(2, 6), (4, 4), (6, 2)\}$, $n = 3$.

$$P(A | B) = 3/18 = 1/6.$$

Section C (answer any eight)

Q4. Solve by the matrix method: $x - y + 2z = 7$, $3x + 4y - 5z = -5$, $2x - y + 3z = 12$.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}, B = [7, -5, 12]^T.$$

$$|A| = 1(12 - 5) + 1(9 + 10) + 2(-3 - 8) = 7 + 19 - 22 = 4 \neq 0.$$

$$\text{Cofactors: } C_{11} = 7, C_{12} = -19, C_{13} = -11; C_{21} = 1, C_{22} = -1, C_{23} = -1; C_{31} = -3, C_{32} = 11, C_{33} = 7.$$

$$\text{adj } A = \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}.$$

$$X = A^{-1}B = (1/4)[7 \cdot 7 + 1 \cdot (-5) - 3 \cdot 12, -19 \cdot 7 - 1 \cdot (-5) + 11 \cdot 12, -11 \cdot 7 - 1 \cdot (-5) + 7 \cdot 12]^T = (1/4)[8, 4, 12]^T.$$

$$x = 2, y = 1, z = 3. \text{ Check in equation 2: } 6 + 4 - 15 = -5.$$

Q5. Prove that $|a - b - c, 2a, 2a; 2b, b - c - a, 2b; 2c, 2c, c - a - b| = (a + b + c)^3$.

$R_1 \rightarrow R_1 + R_2 + R_3$: the first row becomes $(a + b + c, a + b + c, a + b + c)$. Take $(a + b + c)$ out:

$$\Delta = (a + b + c) |1, 1, 1; 2b, b - c - a, 2b; 2c, 2c, c - a - b|.$$

$$C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1:$$

$$\Delta = (a + b + c) |1, 0, 0; 2b, -(a + b + c), 0; 2c, 0, -(a + b + c)|.$$

$$\text{Expand along } R_1: \Delta = (a + b + c) \cdot (a + b + c)^2 = (a + b + c)^3. \text{ Proved.}$$

Q6. If $y = (\sin^{-1}x)^2$, prove that $(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - 2 = 0$.

$$y_1 = 2 \sin^{-1}x \cdot 1/\sqrt{1-x^2} \Rightarrow \sqrt{1-x^2} y_1 = 2 \sin^{-1}x.$$

$$\text{Square: } (1 - x^2) y_1^2 = 4(\sin^{-1}x)^2 = 4y.$$

$$\text{Differentiate: } (1 - x^2) \cdot 2y_1 y_2 - 2x y_1^2 = 4y_1.$$

$$\text{Divide by } 2y_1 \ (y_1 \neq 0): (1 - x^2)y_2 - x y_1 - 2 = 0. \text{ Proved.}$$

Q7. From a square sheet of side 18 cm, equal squares are cut from each corner and the sides are folded up to make an open box. Find the side of the square cut off for the box to have maximum volume, and find that volume.

$$\text{Let the cut side be } x \text{ cm } (0 < x < 9). V = x(18 - 2x)^2.$$

$$\frac{dV}{dx} = (18 - 2x)^2 - 4x(18 - 2x) = (18 - 2x)(18 - 6x) = 0 \Rightarrow x = 3 \ (x = 9 \text{ gives no box}).$$

$$\frac{d^2V}{dx^2} = 24x - 144; \text{ at } x = 3 \text{ it is } -72 < 0 \Rightarrow \text{maximum.}$$

$$\mathbf{x = 3 \text{ cm; } V_{\max} = 3 \times 12 \times 12 = 432 \text{ cm}^3.}$$

Q8. Evaluate $\int_0^\pi x \sin x / (1 + \cos^2 x) dx$.

$$\text{Let } I \text{ be the integral. Use } \int_0^a f(x) dx = \int_0^a f(a-x) dx:$$

$$I = \int_0^\pi (\pi - x) \sin x / (1 + \cos^2 x) dx. \text{ Add: } 2I = \pi \int_0^\pi \sin x / (1 + \cos^2 x) dx.$$

$$\text{Put } t = \cos x, dt = -\sin x dx; x = 0 \rightarrow t = 1, x = \pi \rightarrow t = -1:$$

$$2I = \pi \int_{-1}^1 dt / (1 + t^2) = \pi [\tan^{-1} t]_{-1}^1 = \pi(\pi/4 + \pi/4) = \pi^2/2.$$

$$\mathbf{I = \pi^2/4.}$$

Q9. Find the area of the region bounded by the parabola $y^2 = 4x$ and the line $y = 2x$.

$$\text{Intersection: } (2x)^2 = 4x \Rightarrow 4x(x - 1) = 0 \Rightarrow x = 0, 1; \text{ points } (0, 0) \text{ and } (1, 2).$$

$$\text{For } 0 \leq x \leq 1 \text{ the upper curve is } y = 2\sqrt{x} \text{ and the lower is } y = 2x.$$

$$\text{Area} = \int_0^1 (2\sqrt{x} - 2x) dx = [(4/3)x^{3/2} - x^2]_0^1 = 4/3 - 1 = \mathbf{1/3 \text{ sq. unit.}}$$

Q10. Solve $dy/dx + 2y \tan x = \sin x$, given $y = 0$ when $x = \pi/3$.

$$\text{Linear with } P = 2 \tan x, Q = \sin x. \text{ IF} = e^{\int 2 \tan x dx} = e^{2 \log \sec x} = \sec^2 x.$$

$$y \sec^2 x = \int \sin x \sec^2 x dx = \int \sec x \tan x dx = \sec x + C.$$

$$\text{At } x = \pi/3: 0 = 2 + C \Rightarrow C = -2.$$

$$y \sec^2 x = \sec x - 2 \Rightarrow \mathbf{y = \cos x - 2 \cos^2 x.}$$

Q11. Find the shortest distance between the lines $\mathbf{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$ and $\mathbf{r} = (2\hat{i} - \hat{j} - \hat{k}) + \mu(2\hat{i} + \hat{j} + 2\hat{k})$.

$$\mathbf{a}_1 = (1, 2, 1), \mathbf{b}_1 = (1, -1, 1); \mathbf{a}_2 = (2, -1, -1), \mathbf{b}_2 = (2, 1, 2).$$

$$\mathbf{b}_1 \times \mathbf{b}_2 = \hat{i}(-2 - 1) - \hat{j}(2 - 2) + \hat{k}(1 + 2) = -3\hat{i} + 0\hat{j} + 3\hat{k}; |\mathbf{b}_1 \times \mathbf{b}_2| = 3\sqrt{2}.$$

$$\mathbf{a}_2 - \mathbf{a}_1 = \hat{i} - 3\hat{j} - 2\hat{k}. (\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = -3 + 0 - 6 = -9.$$

$$SD = |-9|/(3\sqrt{2}) = \mathbf{3/\sqrt{2} \text{ units}}.$$

Q12. Solve graphically: maximise $Z = 4x + y$ subject to $x + y \leq 50$, $3x + y \leq 90$, $x \geq 0$, $y \geq 0$.

Lines: $x + y = 50$ through $(50, 0)$, $(0, 50)$; $3x + y = 90$ through $(30, 0)$, $(0, 90)$.

Intersection: subtract $\Rightarrow 2x = 40 \Rightarrow x = 20, y = 30$.

The feasible region is bounded with corners:

Corner	$Z = 4x + y$
$(0, 0)$	0
$(30, 0)$	120
$(20, 30)$	110
$(0, 50)$	50

Max $Z = 120$ at $x = 30, y = 0$.

Q13. Machines A, B and C produce 50%, 30% and 20% of the bolts of a factory; 2%, 3% and 4% of their output respectively is defective. A bolt drawn at random is defective. Find the probability that it was made by machine C.

$$P(E_1) = 0.5, P(E_2) = 0.3, P(E_3) = 0.2; P(D | E_1) = 0.02, P(D | E_2) = 0.03, P(D | E_3) = 0.04.$$

$$P(D) = 0.010 + 0.009 + 0.008 = 0.027 \text{ (total probability).}$$

$$\text{Bayes: } P(E_3 | D) = 0.008/0.027 = \mathbf{8/27}.$$

$$\text{Similarly } P(E_1 | D) = 10/27 \text{ and } P(E_2 | D) = 9/27 = 1/3; \text{ the three add to 1.}$$

MARKING PITFALLS SEEN IN THIS PAPER

Writing $|A|$ without showing the expansion; forgetting the $+C$ in indefinite integrals; not checking that a critical point is a maximum (second derivative); leaving an LPP answer without stating the point; using $P(A | B)$ when the question asks for $P(B | A)$.

MCQ Bank: 104 Chapter-wise Questions

HOW TO PRACTISE MCQS

Cover the answer line, choose an option in under a minute, then check. Eliminate options by substituting simple values ($x = 0$, $x = 1$), checking dimensions or signs, and testing special cases. Circle every question you got wrong and revise that chapter's formula box.

1. Relations and Functions

1. The number of relations on a set with 3 elements is (a) 9 (b) 64 (c) 512 (d) 8

(c). $2^9 = 512$.

2. The number of bijections from $\{1, 2, 3, 4\}$ to itself is (a) 4 (b) 16 (c) 24 (d) 256

(c). $4! = 24$.

3. $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^2$ is (a) one-one onto (b) one-one into (c) many-one onto (d) many-one into

(d). $f(1) = f(-1)$, and negative numbers have no pre-image.

4. If $f(x) = 2x + 3$ and $g(x) = x^2$, then $(g \circ f)(1) =$ (a) 5 (b) 25 (c) 11 (d) 7

(b). $f(1) = 5$, $g(5) = 25$.

5. The identity element for the binary operation $a * b = a + b - 4$ on $\mathbb{Z}$ is (a) 0 (b) 4 (c) -4 (d) 1

(b). $a + e - 4 = a \Rightarrow e = 4$.

6. The smallest equivalence relation on $\{1, 2, 3\}$ has how many ordered pairs? (a) 0 (b) 1 (c) 3 (d) 9

(c). The identity relation $\{(1, 1), (2, 2), (3, 3)\}$.

7. If $f(x) = (3x - 2)/(x + 1)$, $x \neq -1$, then $f^{-1}(x) =$ (a) $(x + 2)/(3 - x)$ (b) $(x - 2)/(3 + x)$ (c) $(3 - x)/(x + 2)$ (d) $(x + 1)/(3x - 2)$

(a). $y(x + 1) = 3x - 2 \Rightarrow x(y - 3) = -2 - y \Rightarrow x = (y + 2)/(3 - y)$.

8. The relation "is perpendicular to" on the set of lines in a plane is (a) reflexive (b) symmetric (c) transitive (d) an equivalence relation

(b). Only symmetric.

2. Inverse Trigonometric Functions

9. The principal value of $\tan^{-1}(-\sqrt{3})$ is (a) $2\pi/3$ (b) $-\pi/3$ (c) $\pi/3$ (d) $-\pi/6$

(b).

10. The domain of $\sin^{-1}(2x)$ is (a) $[-1, 1]$ (b) $[-1/2, 1/2]$ (c) $[0, 1]$ (d) $[-2, 2]$

(b). $-1 \leq 2x \leq 1$.

11. $\sin^{-1}x + \cos^{-1}x =$ (a) 0 (b) π (c) $\pi/2$ (d) $\pi/4$

(c).

12. $\sec^{-1}(2) =$ (a) $\pi/6$ (b) $\pi/3$ (c) $2\pi/3$ (d) $\pi/4$

(b). $\cos^{-1}(1/2)$.

13. $\tan(2 \tan^{-1}(1/5)) =$ (a) $2/5$ (b) $5/12$ (c) $12/5$ (d) $1/5$

(b). $(2/5)/(1 - 1/25) = (2/5)(25/24) = 5/12$.

14. $\cos^{-1}(-1/2) =$ (a) $-\pi/3$ (b) $\pi/3$ (c) $2\pi/3$ (d) $4\pi/3$

(c). $\cos^{-1}(-x) = \pi - \cos^{-1}x$.

15. $\sin(\tan^{-1}(3/4)) =$ (a) $3/5$ (b) $4/5$ (c) $3/4$ (d) $5/3$

(a). Triangle 3-4-5.

16. $\tan^{-1}2 + \tan^{-1}3 =$ (a) $\pi/4$ (b) $3\pi/4$ (c) $\pi/2$ (d) π

(b). Since $2 \cdot 3 > 1$, the sum is $\pi + \tan^{-1}[(5)/(1 - 6)] = \pi - \pi/4$.

3. Matrices

17. The number of 2×2 matrices with entries 0 or 1 is (a) 4 (b) 8 (c) 16 (d) 32

(c). 2^4 .

18. If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $A^2 =$ (a) A (b) I (c) O (d) $-A$

(c).

19. If A is symmetric and B is skew-symmetric of the same order, then $AB - BA$ is (a) symmetric (b) skew-symmetric (c) zero (d) identity

(a). $(AB - BA)^T = B^T A^T - A^T B^T = -BA + AB$.

20. $(AB)^T =$ (a) $A^T B^T$ (b) $B^T A^T$ (c) AB (d) BA

(b).

21. If $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, then $A^3 =$ (a) $\begin{bmatrix} 6 & 0 \\ 0 & 9 \end{bmatrix}$ (b) $\begin{bmatrix} 8 & 0 \\ 0 & 27 \end{bmatrix}$ (c) $\begin{bmatrix} 8 & 0 \\ 0 & 9 \end{bmatrix}$ (d) $6I$

(b). Powers of a diagonal matrix act entry-wise.

22. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, the trace of A is (a) 4 (b) 5 (c) -2 (d) 10

(b). $1 + 4$.

23. If $A^2 = A$, then $(I + A)^2 - 3A =$ (a) I (b) A (c) O (d) $2A$

(a). $I + 2A + A^2 - 3A = I + 3A - 3A$.

24. A matrix that is both symmetric and skew-symmetric is (a) I (b) a diagonal matrix (c) the zero matrix (d) any square matrix

(c). $A = A^T = -A \Rightarrow A = O$.

4. Determinants

25. $|\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}| =$ (a) 5 (b) 11 (c) 8 (d) -11

(b). $8 + 3$.

26. If A is 3×3 and $|A| = 5$, then $|2A| =$ (a) 10 (b) 20 (c) 40 (d) 30

(c). $2^3 \times 5$.

27. If two rows of a determinant are identical, its value is (a) 1 (b) 0 (c) -1 (d) doubled

(b).

28. Area of the triangle with vertices $(0, 0)$, $(4, 0)$, $(0, 3)$ is (a) 12 (b) 6 (c) 7 (d) 3

(b). $(1/2)|4 \cdot 3|$.

29. If A is invertible of order 3, then $|A^{-1}| =$ (a) $|A|$ (b) $1/|A|$ (c) $|A|^2$ (d) $-|A|$

(b).

30. If $|x, 3; 2, x| = |4, 3; 2, 4|$, then $x =$ (a) ± 4 (b) 4 only (c) ± 2 (d) ± 3

(a). $x^2 - 6 = 10$.

31. The system $AX = B$ has a unique solution when (a) $|A| = 0$ (b) $|A| \neq 0$ (c) $B = O$ (d) A is singular

(b).

32. $|1, \omega, \omega^2; \omega, \omega^2, 1; \omega^2, 1, \omega|$ (ω a cube root of unity, $\omega \neq 1$) = (a) 1 (b) 0 (c) ω (d) 3

(b). $C_1 + C_2 + C_3$ gives a column of $1 + \omega + \omega^2 = 0$.

5. Continuity and Differentiability

33. $f(x) = (\sin 3x)/x$ for $x \neq 0$ and $f(0) = k$ is continuous at 0 if $k =$ (a) 1 (b) 3 (c) 0 (d) $1/3$

(b).

34. $d/dx(\tan^{-1}x^2) =$ (a) $2x/(1 + x^4)$ (b) $1/(1 + x^4)$ (c) $2x/(1 + x^2)$ (d) $x/(1 + x^4)$

(a).

35. If $y = \log(\cos e^x)$, then $dy/dx =$ (a) $-e^x \tan e^x$ (b) $\tan e^x$ (c) $-\tan e^x$ (d) $e^x \cot e^x$

(a).

36. If $x = a \cos \theta$, $y = a \sin \theta$, then $dy/dx =$ (a) $\tan \theta$ (b) $-\cot \theta$ (c) $\cot \theta$ (d) $-\tan \theta$

(b). $(a \cos \theta)/(-a \sin \theta)$.

37. If $x^2 + y^2 = 25$, then $dy/dx =$ (a) x/y (b) $-x/y$ (c) y/x (d) $-y/x$

(b).

38. The number of points where $f(x) = |x| + |x - 1|$ is not differentiable is (a) 0 (b) 1 (c) 2 (d) 3

(c). $x = 0$ and $x = 1$.

39. If $y = e^x + e^{-x}$, then $d^2y/dx^2 =$ (a) y (b) $-y$ (c) $2y$ (d) 0

(a).

40. Rolle's theorem for $f(x) = x^2 - 4x$ on $[0, 4]$ gives $c =$ (a) 1 (b) 2 (c) 3 (d) 4

(b). $f'(c) = 2c - 4 = 0$.

6. Applications of Derivatives

41. The side of a square grows at 3 cm/s. Its area grows, when the side is 10 cm, at (a) 30 (b) 60 (c) 100 (d) $6 \text{ cm}^2/\text{s}$

(b). $2 \times 10 \times 3$.

42. The slope of the normal to $y = x^3$ at (1, 1) is (a) 3 (b) -3 (c) $1/3$ (d) $-1/3$

(d). Tangent slope 3.

43. $f(x) = e^x$ on $\mathbb{R}$ is (a) increasing (b) decreasing (c) constant (d) neither

(a).

44. The minimum value of $x^2 - 6x + 10$ is (a) 10 (b) 1 (c) 3 (d) 0

(b). $(x - 3)^2 + 1$.

45. The maximum value of $\sin x + \cos x$ is (a) 1 (b) 2 (c) $\sqrt{2}$ (d) $1/\sqrt{2}$

(c).

46. The tangent to $y = x^2$ is parallel to $y = 4x + 1$ at (a) (1, 1) (b) (2, 4) (c) (4, 16) (d) (-2, 4)

(b). $2x = 4$.

47. The approximate value of $\sqrt{25.2}$ using differentials is (a) 5.02 (b) 5.2 (c) 5.002 (d) 5.04

(a). $5 + 0.2/(2 \cdot 5) = 5.02$.

48. Two positive numbers have sum 20. Their maximum product is (a) 100 (b) 99 (c) 96 (d) 400

(a). 10×10 .

7. Integrals

49. $\int \sec^2 x \, dx =$ (a) $\tan x + C$ (b) $\sec x + C$ (c) $-\cot x + C$ (d) $\sec x \tan x + C$

(a).

50. $\int e^x(\sin x + \cos x) \, dx =$ (a) $e^x \cos x + C$ (b) $e^x \sin x + C$ (c) $-e^x \cos x + C$ (d) $e^x(\sin x - \cos x) + C$

(b). $\int e^x[f + f'] \, dx = e^x f$.

51. $\int dx/(x^2 + 9) =$ (a) $\tan^{-1}(x/3) + C$ (b) $(1/3)\tan^{-1}(x/3) + C$ (c) $3 \tan^{-1}(x/3) + C$ (d) $(1/9)\tan^{-1}x + C$

(b).

52. $\int_0^1 x e^x \, dx =$ (a) e (b) 1 (c) $e - 1$ (d) 0

(b). $[(x - 1)e^x]_0^1 = 0 - (-1)$.

53. $\int_0^{\pi/2} \sin x/(\sin x + \cos x) \, dx =$ (a) $\pi/2$ (b) $\pi/4$ (c) π (d) 0

(b). Use the $a - x$ property and add.

54. $\int dx/\sqrt{4 - x^2} =$ (a) $\sin^{-1}(x/2) + C$ (b) $\cos^{-1}(x/2) + C$ (c) $(1/2)\sin^{-1}(x/2) + C$ (d) $\log|x + \sqrt{4 - x^2}| + C$

(a).

55. $\int_{-2}^2 |x| \, dx =$ (a) 0 (b) 2 (c) 4 (d) 8

(c). $2 \int_0^2 x \, dx = 4$.

56. $\int \tan x \, dx =$ (a) $\log|\cos x| + C$ (b) $\log|\sec x| + C$ (c) $\sec^2 x + C$ (d) $\log|\sin x| + C$

(b).

8. Applications of Integrals

57. The area bounded by $y = x^2$, the x-axis and $x = 3$ is (a) 3 (b) 9 (c) 27 (d) 6

(b). $x^3/3$ from 0 to 3.

58. The area enclosed by the circle $x^2 + y^2 = 16$ is (a) 4π (b) 8π (c) 16π (d) 64π

(c).

59. The area enclosed by $x^2/9 + y^2/4 = 1$ is (a) 6π (b) 13π (c) 36π (d) 3π

(a). $\pi ab = \pi \cdot 3 \cdot 2$.

60. The area between $y = x$ and $y = x^2$ is (a) $1/2$ (b) $1/3$ (c) $1/6$ (d) 1

(c). $\int_0^1 (x - x^2) dx$.

61. The area under $y = \sin x$ from 0 to π is (a) 1 (b) 2 (c) 0 (d) π

(b).

62. The area bounded by $y = \cos x$ and the x-axis from 0 to π is (a) 0 (b) 1 (c) 2 (d) π

(c). Take $|f|$ on each part: $1 + 1$.

63. The area bounded by $y^2 = 4ax$ and its latus rectum $x = a$ is (a) $4a^2/3$ (b) $8a^2/3$ (c) $2a^2/3$ (d) $16a^2/3$

(b). $2 \int_0^a 2\sqrt{ax} \, dx = 8a^2/3$.

64. The area bounded by $y = 2x$, the x-axis and $x = 2$ is (a) 2 (b) 4 (c) 8 (d) 1

(b). Triangle $(1/2)(2)(4)$.

9. Differential Equations

65. The order and degree of $(d^2y/dx^2)^3 + (dy/dx)^4 = x$ are (a) 2, 3 (b) 2, 4 (c) 3, 2 (d) 4, 2

(a).

66. The degree of $y'' + \sin(y') = 0$ is (a) 1 (b) 2 (c) 0 (d) not defined

(d). Not a polynomial in y' .

67. The general solution of $dy/dx = e^x + y$ is (a) $e^x + e^{-y} = C$ (b) $e^x + e^y = C$ (c) $e^{-x} + e^y = C$ (d) $e^x - e^y = C$

(a). $e^{-y} dy = e^x dx \Rightarrow -e^{-y} = e^x - C$.

68. The integrating factor of $dy/dx - y = e^x$ is (a) e^x (b) e^{-x} (c) x (d) $-x$

(b).

69. The number of arbitrary constants in the general solution of a 3rd order DE is (a) 0 (b) 1 (c) 2 (d) 3

(d).

70. The solution of $dy/dx = y/x$ is (a) $y = Cx$ (b) $xy = C$ (c) $y = x + C$ (d) $y^2 = Cx$

(a).

71. A homogeneous DE $dy/dx = F(y/x)$ is solved by putting (a) $y = vx$ (b) $y = v + x$ (c) $x = y + v$ (d) $y = x^v$

(a).

72. The DE of the family $y = A e^{2x}$ is (a) $y' = 2y$ (b) $y' = y$ (c) $y'' = 2y$ (d) $y' = -2y$

(a).

10. Vector Algebra

73. $|3\hat{i} - 4\hat{j}| =$ (a) 7 (b) 1 (c) 5 (d) 25

(c).

74. $\hat{i} \cdot (\hat{j} \times \hat{k}) =$ (a) 0 (b) 1 (c) -1 (d) 3

(b).

75. If $\mathbf{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{j} + 3\hat{k}$ are perpendicular, $\lambda =$ (a) 1 (b) $5/2$ (c) 2 (d) $-5/2$

(b). $2 - 2\lambda + 3 = 0$.

76. The area of the parallelogram with adjacent sides $\hat{i}$ and $\hat{j} + \hat{k}$ is (a) 1 (b) $\sqrt{2}$ (c) 2 (d) 0

(b). $\hat{i} \times (\hat{j} + \hat{k}) = \hat{k} - \hat{j}$.

77. If $|\mathbf{a} \times \mathbf{b}| = 4$ and $\mathbf{a} \cdot \mathbf{b} = 3$, then $|\mathbf{a}|^2|\mathbf{b}|^2 =$ (a) 7 (b) 12 (c) 25 (d) 1

(c). Lagrange's identity: $16 + 9$.

78. The unit vector along $\hat{i} + \hat{j} + \hat{k}$ is (a) $\hat{i} + \hat{j} + \hat{k}$ (b) $(\hat{i} + \hat{j} + \hat{k})/3$ (c) $(\hat{i} + \hat{j} + \hat{k})/\sqrt{3}$ (d) $(\hat{i} + \hat{j})/\sqrt{2}$

(c).

79. Vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are coplanar if (a) $\mathbf{a} \cdot \mathbf{b} = 0$ (b) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = 0$ (c) $\mathbf{a} \times \mathbf{b} = 0$ (d) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = 1$

(b).

80. The position vector of the midpoint of $\mathbf{A}(2\hat{i} + 3\hat{j})$ and $\mathbf{B}(4\hat{i} - \hat{j})$ is (a) $3\hat{i} + \hat{j}$ (b) $6\hat{i} + 2\hat{j}$ (c) $\hat{i} - 2\hat{j}$ (d) $3\hat{i} + 2\hat{j}$

(a).

11. Three-dimensional Geometry

81. The direction ratios of the line $(x - 1)/2 = (y + 3)/(-1) = (z - 4)/5$ are (a) 1, -3, 4 (b) 2, -1, 5 (c) -1, 3, -4 (d) 2, 1, 5

(b).

82. The distance of (2, 3, 4) from the plane $3x - 6y + 2z + 11 = 0$ is (a) 1 (b) 2 (c) 3 (d) 7

(a). $|6 - 18 + 8 + 11|/7 = 7/7$.

83. The planes $2x + y - z = 3$ and $x - y + z = 5$ are (a) parallel (b) perpendicular (c) inclined at 60° (d) coincident

(b). Normals (2, 1, -1) and (1, -1, 1): $2 - 1 - 1 = 0$.

84. The angle between the lines with direction ratios 1, 1, 0 and 0, 1, 1 is (a) 30° (b) 45° (c) 60° (d) 90°

(c). $\cos \theta = 1/(\sqrt{2} \cdot \sqrt{2}) = 1/2$.

85. The equation of the plane with intercepts 2, 3, 4 on the axes is (a) $x/2 + y/3 + z/4 = 1$ (b) $2x + 3y + 4z = 1$ (c) $x/2 + y/3 + z/4 = 0$ (d) $6x + 4y + 3z = 1$

(a).

86. The direction cosines of the z-axis are (a) 1, 0, 0 (b) 0, 1, 0 (c) 0, 0, 1 (d) 1, 1, 1

(c).

87. The distance between the parallel planes $x + 2y + 2z = 3$ and $x + 2y + 2z = 9$ is (a) 6 (b) 2 (c) 3 (d) 1

(b). $|9 - 3|/3$.

88. If l , m , n are direction cosines of a line, then $l^2 + m^2 + n^2 =$ (a) 0 (b) 1 (c) 3 (d) -1

(b).

12. Linear Programming

89. In an LPP the optimal value, if it exists for a bounded region, occurs at (a) an interior point (b) a corner point (c) the origin always (d) any boundary point only

(b). Corner point theorem.

90. The maximum of $Z = x + y$ subject to $x + y \leq 4$, $x, y \geq 0$ is (a) 2 (b) 4 (c) 8 (d) 0

(b). Every point on the segment from (4, 0) to (0, 4) gives 4.

91. The point (1, 2) lies in the region $x + y \leq 4$, $x \geq 0$, $y \geq 0$? (a) yes (b) no (c) on the boundary only (d) cannot say

(a). $1 + 2 = 3 \leq 4$.

92. Corner points of a feasible region are (0, 0), (5, 0), (3, 4), (0, 5). Max of $Z = 2x + 3y$ is (a) 10 (b) 15 (c) 18 (d) 17

(c). Values 0, 10, 18, 15.

93. The constraints $x \geq 0$, $y \geq 0$ are called (a) objective constraints (b) non-negativity constraints (c) optimal conditions (d) slack conditions

(b).

94. For corners (0, 2), (3, 0), (6, 0), (6, 8), (0, 5), the minimum of $Z = 4x + 6y$ is (a) 0 (b) 12 (c) 30 (d) 72

(b). Values 12, 12, 24, 72, 30; minimum 12 along the segment from (0, 2) to (3, 0).

95. If the feasible region is unbounded, a maximum of Z (a) always exists (b) may not exist (c) never exists (d) is always zero

(b).

96. The half-plane $2x + 3y \leq 6$ contains (a) the origin (b) (3, 3) (c) (0, 3) (d) (4, 0)

(a). $0 \leq 6$.

13. Probability

97. If $P(A) = 0.5$, $P(B) = 0.4$ and A, B are independent, $P(A \cup B) =$ (a) 0.9 (b) 0.7 (c) 0.2 (d) 0.1

(b). $0.5 + 0.4 - 0.2$.

98. A coin is tossed 3 times. $P(\text{exactly 2 heads}) =$ (a) $1/8$ (b) $3/8$ (c) $1/2$ (d) $1/4$

(b). ${}^3C_2/8$.

99. The mean of the binomial distribution $B(10, 0.4)$ is (a) 4 (b) 2.4 (c) 6 (d) 10

(a). np .

100. The variance of $B(10, 0.4)$ is (a) 4 (b) 2.4 (c) 1.6 (d) 0.24

(b). $npq = 10(0.4)(0.6)$.

101. If $P(A) = 0.3$ and $P(B | A) = 0.5$, then $P(A \cap B) =$ (a) 0.8 (b) 0.15 (c) 0.6 (d) 0.2

(b).

102. For a probability distribution $X = 0, 1, 2$ with $P = k, 2k, 3k$, $k =$ (a) $1/3$ (b) $1/6$ (c) $1/2$ (d) 1

(b). $6k = 1$.

103. $E(X)$ for the distribution in Q102 is (a) $4/3$ (b) 1 (c) 2 (d) $5/3$

(a). $0 + 2/6 + 6/6 = 8/6 = 4/3$.

104. Two cards are drawn from 52 without replacement. $P(\text{both are kings}) =$ (a) $1/221$ (b) $1/169$ (c) $4/663$ (d) $1/13$

(a). $(4/52)(3/51) = 12/2652 = 1/221$.

CHECK YOUR SCORE

90 or above: excellent. 75 to 89: revise the chapters where you lost two or more.

Below 75: re-read the chapter notes and formula boxes, then reattempt this bank after a week.

HOTS and Extra Worked Examples

WHAT HOTS QUESTIONS TEST

Higher Order Thinking Skills questions combine two ideas, need a clever substitution, or ask you to prove a general result. Read the question twice, write down what is given and what is required, and look for a standard pattern hidden inside it.

Relations, Functions and Inverse Trigonometry

1. Let $f: \mathbb{R} - \{2/3\} \rightarrow \mathbb{R}$, $f(x) = (4x + 3)/(6x - 4)$. Show that $f \circ f(x) = x$ and hence find f^{-1} .

$$f(f(x)) = [4(4x + 3)/(6x - 4) + 3]/[6(4x + 3)/(6x - 4) - 4] = [16x + 12 + 18x - 12]/[24x + 18 - 24x + 16] = 34x/34 = x.$$

So $f \circ f = I$, which means f is invertible and $f^{-1} = f$, i.e. $f^{-1}(x) = (4x + 3)/(6x - 4)$.

2. Solve $\tan^{-1}[(x - 1)/(x - 2)] + \tan^{-1}[(x + 1)/(x + 2)] = \pi/4$.

Take tan of both sides: $[(x - 1)/(x - 2) + (x + 1)/(x + 2)]/[1 - (x^2 - 1)/(x^2 - 4)] = 1$.

Numerator: $[(x - 1)(x + 2) + (x + 1)(x - 2)]/(x^2 - 4) = (2x^2 - 4)/(x^2 - 4)$.

Denominator: $[(x^2 - 4) - (x^2 - 1)]/(x^2 - 4) = -3/(x^2 - 4)$.

So $(2x^2 - 4)/(-3) = 1 \Rightarrow 2x^2 = 1 \Rightarrow x = \pm 1/\sqrt{2}$. Both values satisfy the original equation (check numerically).

Matrices and Determinants

3. For $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, show that $A^2 - 4A + I = O$ and use it to find A^{-1} .

$$A^2 = \begin{bmatrix} 4 + 3 & 6 + 6 \\ 2 + 2 & 3 + 4 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}. 4A = \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix}.$$

$$A^2 - 4A + I = \begin{bmatrix} 7 - 8 + 1 & 0 \\ 0 & 7 - 8 + 1 \end{bmatrix} = O.$$

$$\text{Multiply by } A^{-1}: A - 4I + A^{-1} = O \Rightarrow A^{-1} = 4I - A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}.$$

4. Prove that $|1, a, a^2; 1, b, b^2; 1, c, c^2| = (a - b)(b - c)(c - a)$.

$$R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3:$$

$$\Delta = |0, a - b, a^2 - b^2; 0, b - c, b^2 - c^2; 1, c, c^2| = (a - b)(b - c)|0, 1, a + b; 0, 1, b + c; 1, c, c^2|.$$

$$\text{Expand along } C_1: \Delta = (a - b)(b - c)[(b + c) - (a + b)] = (a - b)(b - c)(c - a). \text{ Proved.}$$

Continuity, Differentiability and Derivatives

5. Find k so that $f(x) = (1 - \cos 4x)/(8x^2)$ for $x \neq 0$ and $f(0) = k$ is continuous at $x = 0$.

$$1 - \cos 4x = 2 \sin^2 2x. \lim_{x \rightarrow 0} (2 \sin^2 2x)/(8x^2) = (1/4) \lim_{x \rightarrow 0} (\sin 2x/x)^2 = (1/4)(2)^2 = 1. \mathbf{k = 1}.$$

6. If $\sqrt{(1-x^2)} + \sqrt{(1-y^2)} = a(x-y)$, prove that $dy/dx = \sqrt{(1-y^2)/(1-x^2)}$.

Put $x = \sin A$, $y = \sin B$: $\cos A + \cos B = a(\sin A - \sin B)$.

$2 \cos[(A+B)/2] \cos[(A-B)/2] = 2a \cos[(A+B)/2] \sin[(A-B)/2] \Rightarrow \cot[(A-B)/2] = a$.

So $A - B = 2 \cot^{-1}a$, i.e. $\sin^{-1}x - \sin^{-1}y = \text{constant}$.

Differentiate: $1/\sqrt{(1-x^2)} - (1/\sqrt{(1-y^2)}) dy/dx = 0 \Rightarrow dy/dx = \sqrt{(1-y^2)/(1-x^2)}$.

Proved.

7. If $y = e^{a \sin^{-1}x}$, prove that $(1-x^2)y_2 - x y_1 - a^2y = 0$.

$y_1 = a y/\sqrt{(1-x^2)} \Rightarrow (1-x^2)y_1^2 = a^2y^2$.

Differentiate: $(1-x^2)2y_1y_2 - 2xy_1^2 = 2a^2y y_1$. Divide by $2y_1$: $(1-x^2)y_2 - x y_1 - a^2y = 0$. Proved.

8. Show that of all rectangles inscribed in a circle of radius r , the square has the maximum area.

Let a diagonal (a diameter, length $2r$) make angle θ with one side. Sides: $2r \cos \theta$ and $2r \sin \theta$.

Area $A = 4r^2 \sin \theta \cos \theta = 2r^2 \sin 2\theta$, $0 < \theta < \pi/2$.

$dA/d\theta = 4r^2 \cos 2\theta = 0 \Rightarrow \theta = \pi/4$; $d^2A/d\theta^2 = -8r^2 \sin 2\theta < 0$ there $\Rightarrow$ maximum.

At $\theta = \pi/4$ both sides equal $r\sqrt{2}$, so the rectangle is a square with **maximum area $2r^2$** .

9. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangent is parallel to the x-axis.

$2x + 2y y' - 2 = 0 \Rightarrow y' = (1-x)/y$. $y' = 0 \Rightarrow x = 1$. Then $1 + y^2 - 2 - 3 = 0 \Rightarrow y = \pm 2$.
(1, 2) and (1, -2).

Integrals

10. Evaluate $\int dx/[x(x^4 + 1)]$.

Multiply top and bottom by x^3 : $\int x^3 dx/[x^4(x^4 + 1)]$. Put $t = x^4$, $dt = 4x^3 dx$:

$(1/4)\int dt/[t(t+1)] = (1/4)\int[1/t - 1/(t+1)] dt = (1/4)\log|t/(t+1)| + C$.

$(1/4) \log[x^4/(x^4 + 1)] + C$.

11. Evaluate $\int (x^2 + 1)/(x^4 + 1) dx$.

Divide by x^2 : $\int (1 + 1/x^2)/(x^2 + 1/x^2) dx$. Put $t = x - 1/x$, $dt = (1 + 1/x^2)dx$ and $x^2 + 1/x^2 = t^2 + 2$.

$\int dt/(t^2 + 2) = (1/\sqrt{2}) \tan^{-1}(t/\sqrt{2}) + C = \mathbf{(1/\sqrt{2}) \tan^{-1}[(x^2 - 1)/(\sqrt{2} x)] + C}$.

12. Evaluate $\int_0^{\pi/2} \log \sin x \, dx$.

$$I = \int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \cos x \, dx \text{ (a - x property).}$$

$$2I = \int_0^{\pi/2} \log(\sin x \cos x) \, dx = \int_0^{\pi/2} \log \sin 2x \, dx - (\pi/2) \log 2.$$

$$\text{Put } 2x = t: \int_0^{\pi/2} \log \sin 2x \, dx = (1/2) \int_0^{\pi} \log \sin t \, dt = (1/2) \cdot 2 \int_0^{\pi/2} \log \sin t \, dt = I.$$

$$\text{So } 2I = I - (\pi/2) \log 2 \Rightarrow \mathbf{I = -(\pi/2) \log 2}.$$

13. Evaluate $\int_0^1 \log(1+x)/(1+x^2) \, dx$.

$$\text{Put } x = \tan \theta, dx = \sec^2 \theta \, d\theta; \text{ limits } 0 \text{ to } \pi/4: I = \int_0^{\pi/4} \log(1 + \tan \theta) \, d\theta.$$

$$\text{Replace } \theta \text{ by } \pi/4 - \theta: 1 + \tan(\pi/4 - \theta) = 1 + (1 - \tan \theta)/(1 + \tan \theta) = 2/(1 + \tan \theta).$$

$$I = \int_0^{\pi/4} [\log 2 - \log(1 + \tan \theta)] \, d\theta = (\pi/4) \log 2 - I \Rightarrow \mathbf{I = (\pi/8) \log 2}.$$

14. Find the area of the smaller region bounded by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$.

The line meets the circle at (2, 0) and (0, 2). For $0 \leq x \leq 2$ the circle is above the line.

$$\text{Area} = \int_0^2 \sqrt{4 - x^2} \, dx - \int_0^2 (2 - x) \, dx = \pi - 2.$$

(The first integral is a quarter circle of radius 2, i.e. π ; the second is a triangle of area 2.) **$(\pi - 2)$ sq. units.**

Differential Equations**15. Solve $(x^2 - y^2) \, dx + 2xy \, dy = 0$.**

$$dy/dx = (y^2 - x^2)/(2xy), \text{ homogeneous. Put } y = vx: v + x \, dv/dx = (v^2 - 1)/(2v).$$

$$x \, dv/dx = -(1 + v^2)/(2v) \Rightarrow 2v \, dv/(1 + v^2) = -dx/x.$$

$$\log(1 + v^2) = -\log x + \log C \Rightarrow x(1 + y^2/x^2) = C \Rightarrow \mathbf{x^2 + y^2 = Cx}.$$

16. Form the differential equation of the family of circles touching the y-axis at the origin.

$$\text{Centre } (a, 0), \text{ radius } a: x^2 + y^2 = 2ax. \text{ Differentiate: } 2x + 2y y' = 2a.$$

$$\text{Eliminate } a: x^2 + y^2 = x(2x + 2y y') \Rightarrow \mathbf{2xy \, dy/dx = y^2 - x^2}. \text{ (Compare with example 15: these circles are its solution curves.)}$$

Vectors and Three-dimensional Geometry**17. If $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ with $|\mathbf{a}| = 3, |\mathbf{b}| = 5, |\mathbf{c}| = 7$, find the angle between $\mathbf{a}$ and $\mathbf{b}$.**

$$\mathbf{a} + \mathbf{b} = -\mathbf{c} \Rightarrow |\mathbf{a} + \mathbf{b}|^2 = 49 \Rightarrow 9 + 25 + 2\mathbf{a} \cdot \mathbf{b} = 49 \Rightarrow \mathbf{a} \cdot \mathbf{b} = 15/2}.$$

$$\cos \theta = (15/2)/(3 \times 5) = 1/2 \Rightarrow \mathbf{\theta = \pi/3 (60^\circ)}.$$

18. Find the image of the point (1, 6, 3) in the line $x/1 = (y - 1)/2 = (z - 2)/3$.

General point of the line: $M(\lambda, 1 + 2\lambda, 2 + 3\lambda)$. $PM = (\lambda - 1, 2\lambda - 5, 3\lambda - 1)$.

$PM \perp$ line: $(\lambda - 1) + 2(2\lambda - 5) + 3(3\lambda - 1) = 0 \Rightarrow 14\lambda = 14 \Rightarrow \lambda = 1$. Foot $M = (1, 3, 5)$.

M is the midpoint of P and its image Q : $Q = 2M - P = \mathbf{(1, 0, 7)}$.

19. Find the plane through the line of intersection of $x + y + z = 1$ and $2x + 3y + 4z = 5$ that is perpendicular to $x - y + z = 0$.

Family: $(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0$, normal $(1 + 2\lambda, 1 + 3\lambda, 1 + 4\lambda)$.

Perpendicular to $(1, -1, 1)$: $(1 + 2\lambda) - (1 + 3\lambda) + (1 + 4\lambda) = 0 \Rightarrow 1 + 3\lambda = 0 \Rightarrow \lambda = -1/3$.

Plane: $(1/3)x + 0 \cdot y - (1/3)z + 2/3 = 0 \Rightarrow \mathbf{x - z + 2 = 0}$. Check: $(-2, 3, 0)$ lies on both given planes and on this one.

Probability**20. How many times must a fair coin be tossed so that the probability of getting at least one head is more than 90%?**

$P(\text{at least one head}) = 1 - (1/2)^n > 0.9 \Rightarrow (1/2)^n < 0.1 \Rightarrow 2^n > 10$. $2^3 = 8$, $2^4 = 16$. **At least 4 tosses.**

21. A man speaks the truth 4 times out of 5. He throws a die and reports that it is a six. Find the probability that it is actually a six.

E_1 : six occurs, $P = 1/6$; E_2 : no six, $P = 5/6$. A : he reports six.

$P(A | E_1) = 4/5$ (truth); $P(A | E_2) = 1/5$ (lie).

$P(E_1 | A) = (1/6)(4/5) / [(1/6)(4/5) + (5/6)(1/5)] = (4/30) / (9/30) = \mathbf{4/9}$.

22. Find the probability distribution of the number of doublets in three throws of a pair of dice, and its mean.

$P(\text{doublet}) = 6/36 = 1/6$, $q = 5/6$, $n = 3$; $X \sim B(3, 1/6)$.

X	0	1	2	3
$P(X)$	125/216	75/216	15/216	1/216

Total = $216/216 = 1$. **Mean = $np = 1/2$.**

PATTERNS TO REMEMBER FROM THIS PAGE

$f \circ f = I$ means f is its own inverse. A matrix satisfying a quadratic gives A^{-1} directly.

Substitutions $x = \sin A$ turn root expressions into trigonometry. The $a - x$ property plus adding is the key to most "tricky" definite integrals. Image of a point = $2 \times$ foot - point.

Formula Booklet: Last-day Revision

HOW TO USE THIS BOOKLET

Read one chapter block aloud, close the page and write the formulas from memory. Anything you miss goes on a sticky note for the next morning.

Algebra

Topic	Key results
Relations	Number of relations on an n -element set = 2^{n^2} ; reflexive relations = $2^{n^2 - n}$; equivalence = reflexive + symmetric + transitive.
Functions	From m -set to n -set: n^m functions; one-one functions ($m \leq n$): ${}^n P_m$; bijections of an n -set: $n!$. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
Binary operations	Number on an n -set = n^{n^2} ; identity e : $a * e = e * a = a$; inverse b : $a * b = b * a = e$.
Principal ranges	$\sin^{-1}$: $[-\pi/2, \pi/2]$; $\cos^{-1}$: $[0, \pi]$; $\tan^{-1}$: $(-\pi/2, \pi/2)$; $\cot^{-1}$: $(0, \pi)$; $\sec^{-1}$: $[0, \pi] - \{\pi/2\}$; $\operatorname{cosec}^{-1}$: $[-\pi/2, \pi/2] - \{0\}$.
Inverse trig identities	$\sin^{-1} x + \cos^{-1} x = \pi/2$; $\tan^{-1} x + \cot^{-1} x = \pi/2$; $\sin^{-1}(-x) = -\sin^{-1} x$; $\cos^{-1}(-x) = \pi - \cos^{-1} x$; $\tan^{-1} x \pm \tan^{-1} y = \tan^{-1}[(x \pm y)/(1 \mp xy)]$; $2\tan^{-1} x = \sin^{-1}[2x/(1 + x^2)] = \cos^{-1}[(1 - x^2)/(1 + x^2)] = \tan^{-1}[2x/(1 - x^2)]$ ($ x < 1$).
Matrices	$(AB)^T = B^T A^T$; $A = (A + A^T)/2 + (A - A^T)/2$; $(AB)^{-1} = B^{-1} A^{-1}$; $A^{-1} = \operatorname{adj} A / A $.
Determinants	$ kA = k^n A $; $ AB = A B $; $ \operatorname{adj} A = A ^{n-1}$; $A(\operatorname{adj} A) = A I$; area of triangle = $(1/2) x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) $.
Linear systems	$ A \neq 0$: unique solution $X = A^{-1}B$. $ A = 0$ and $(\operatorname{adj} A)B \neq O$: inconsistent. $ A = 0$ and $(\operatorname{adj} A)B = O$: infinitely many or none (check further).

Calculus

Topic	Key results
Standard limits	$\lim \sin x/x = 1$; $\lim (1 - \cos x)/x^2 = 1/2$; $\lim (e^x - 1)/x = 1$; $\lim \log(1 + x)/x = 1$ (all as $x \rightarrow 0$).
Derivatives	$(x^n)' = nx^{n-1}$; $(e^x)' = e^x$; $(a^x)' = a^x \log a$; $(\log x)' = 1/x$; $(\sin x)' = \cos x$; $(\cos x)' = -\sin x$; $(\tan x)' = \sec^2 x$; $(\cot x)' = -\operatorname{cosec}^2 x$; $(\sec x)' = \sec x \tan x$; $(\operatorname{cosec} x)' = -\operatorname{cosec} x \cot x$.
Inverse trig derivatives	$(\sin^{-1} x)' = 1/\sqrt{1 - x^2}$; $(\cos^{-1} x)' = -1/\sqrt{1 - x^2}$; $(\tan^{-1} x)' = 1/(1 + x^2)$; $(\cot^{-1} x)' = -1/(1 + x^2)$; $(\sec^{-1} x)' = 1/(x \sqrt{x^2 - 1})$.
Mean value theorems	Rolle: $f(a) = f(b) \Rightarrow f'(c) = 0$ for some c in (a, b) . LMVT: $f'(c) = [f(b) - f(a)]/(b - a)$.
Applications	Tangent: $y - y_1 = m(x - x_1)$, $m = dy/dx$ at the point; normal slope $-1/m$. $\Delta y \approx f'(x)\Delta x$. Local max: $f' = 0$, $f'' < 0$; local min: $f' = 0$, $f'' > 0$.
Standard integrals	$\int dx/(x^2 + a^2) = (1/a)\tan^{-1}(x/a)$; $\int dx/(x^2 - a^2) = (1/2a)\log (x - a)/(x + a) $; $\int dx/(a^2 - x^2) = (1/2a)\log (a + x)/(a - x) $; $\int dx/\sqrt{a^2 - x^2} = \sin^{-1}(x/a)$; $\int dx/\sqrt{x^2 \pm a^2} = \log x + \sqrt{x^2 \pm a^2} $ (all $+ C$).
More integrals	$\int \sqrt{a^2 - x^2} dx = (x/2)\sqrt{a^2 - x^2} + (a^2/2)\sin^{-1}(x/a)$; $\int \sqrt{x^2 \pm a^2} dx = (x/2)\sqrt{x^2 \pm a^2} \pm (a^2/2)\log x + \sqrt{x^2 \pm a^2} $; $\int e^x[f(x) + f'(x)]dx = e^x f(x)$ (all $+ C$).
By parts	$\int u v dx = u \int v dx - \int [u' \int v dx] dx$; choose u by ILATE (Inverse, Log, Algebraic, Trig, Exponential).
Definite integral properties	$\int_0^a f(x) dx = \int_0^a f(a - x) dx$; $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$; $\int_{-a}^a f = 2 \int_0^a f$ (f even), 0 (f odd); $\int_0^{2a} f = 2 \int_0^a f$ if $f(2a - x) = f(x)$, 0 if $f(2a - x) = -f(x)$.
Area	Area = $\int_a^b y dx$ or $\int_c^d x dy$; between curves $\int (y_{\text{upper}} - y_{\text{lower}}) dx$. Circle πr^2 ; ellipse πab .
Differential equations	Variable separable: $\int g(y) dy = \int f(x) dx$. Homogeneous: $y = vx$. Linear $dy/dx + P_y y = Q$: IF = $e^{\int P dy}$, $y \cdot \text{IF} = \int Q \cdot \text{IF} dx + C$. For $dx/dy + P_1 x = Q_1$: IF = $e^{\int P_1 dy}$.

Vectors, 3D, LPP and Probability

Topic	Key results
Vectors	$a \cdot b = a b \cos \theta$; $ a \times b = a b \sin \theta$; projection of a on $b = a \cdot b/ b $; area of parallelogram $ a \times b $, triangle $(1/2) a \times b $; $[a \ b \ c] = a \cdot (b \times c)$, volume of parallelepiped; coplanar $\Leftrightarrow [a \ b \ c] = 0$; section formula $(mb + na)/(m + n)$.
Lines	$r = a + \lambda b$; $(x - x_1)/a = (y - y_1)/b = (z - z_1)/c$; angle $\cos \theta = b_1 \cdot b_2 /(b_1 b_2)$; skew lines $SD = (a_2 - a_1) \cdot (b_1 \times b_2) / b_1 \times b_2 $; parallel lines $SD = b \times (a_2 - a_1) / b $.
Planes	$r \cdot n = d$; $ax + by + cz + d = 0$; intercept form $x/a + y/b + z/c = 1$; through 3 points: determinant = 0; distance of $(x_1, y_1, z_1) = ax_1 + by_1 + cz_1 + d /\sqrt{a^2 + b^2 + c^2}$; through intersection: $P_1 + \lambda P_2 = 0$.
LPP	Draw constraints, shade feasible region, list corner points, evaluate Z at each. For an unbounded region, verify the optimum with an extra half-plane test.
Probability	$P(A B) = P(A \cap B)/P(B)$; independent: $P(A \cap B) = P(A)P(B)$; total probability $P(A) = \sum P(E_i)P(A E_i)$; Bayes $P(E_i A) = P(E_i)P(A E_i)/\sum P(E_j)P(A E_j)$.
Random variable	Mean $\mu = \sum x p$; Var $= \sum x^2 p - \mu^2$. Binomial: $P(X = r) = {}^n C_r p^r q^{n-r}$, mean np , variance npq .

Chapter Worksheets with Answers

INSTRUCTIONS

Each worksheet takes about 15 to 20 minutes. Attempt all questions before checking the answer key below each one.

Worksheet 1: Relations and Functions

1. If $f(x) = x^2 + 1$ and $g(x) = 2x - 1$, find $f \circ g$ and $g \circ f$.

$$f \circ g(x) = (2x - 1)^2 + 1 = 4x^2 - 4x + 2; g \circ f(x) = 2x^2 + 1.$$

2. On $\mathbb{Q}^+$, $a * b = ab/3$. Find the identity and the inverse of a .

$$ae/3 = a \Rightarrow e = 3; ab/3 = 3 \Rightarrow b = 9/a.$$

3. How many reflexive relations are there on a set with 3 elements?

$$2^9 - 3 = 2^6 = 64.$$

4. Find the inverse of $f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = (2x + 1)/3$.

$$f^{-1}(x) = (3x - 1)/2.$$

Worksheet 2: Inverse Trigonometric Functions

1. Find $\sin^{-1}(\sin 2\pi/3)$.

$$\pi/3, \text{ since } \sin 2\pi/3 = \sin \pi/3 \text{ and } \pi/3 \in [-\pi/2, \pi/2].$$

2. Show that $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$.

$$\tan^{-1}2 + \tan^{-1}3 = \pi + \tan^{-1}[5/(1 - 6)] = \pi - \pi/4 = 3\pi/4. \text{ Add } \pi/4 \text{ to get } \pi.$$

3. Simplify $\tan^{-1}\{\sqrt{(1 + x^2)} - 1\}/x$, $x \neq 0$.

$$\text{Put } x = \tan \theta: (\sec \theta - 1)/\tan \theta = (1 - \cos \theta)/\sin \theta = \tan(\theta/2). \text{ Answer } (1/2)\tan^{-1}x.$$

4. Solve $2 \tan^{-1}(\cos x) = \tan^{-1}(2 \operatorname{cosec} x)$, $0 < x < \pi/2$.

$$2\cos x/(1 - \cos^2 x) = 2/\sin x \Rightarrow \cos x \sin x = \sin^2 x \Rightarrow \tan x = 1 \Rightarrow x = \pi/4.$$

Worksheet 3: Matrices

1. For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, find AB and BA . Are they equal?

$$AB = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}; BA = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}. \text{ Not equal.}$$

2. Express $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ as the sum of a symmetric and a skew-symmetric matrix.

$$P = (A + A^T)/2 = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix}; Q = (A - A^T)/2 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

3. If $A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$ and $A + A^T = I$, find α ($0 < \alpha < \pi/2$).

$$2\cos \alpha = 1 \Rightarrow \alpha = \pi/3.$$

4. Find the inverse of $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$ by elementary row operations.

$$\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix} \text{ (determinant 1).}$$

Worksheet 4: Determinants

1. Evaluate $|\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}|$.

$$0 \text{ (} R_3 - R_2 = R_2 - R_1 \text{).}$$

2. Find the area of the triangle with vertices $(1, 0)$, $(6, 0)$, $(4, 3)$.

$$(1/2) \times 5 \times 3 = 15/2 \text{ sq. units.}$$

3. Find k if the area of the triangle with vertices (k, 0), (4, 0), (0, 2) is 4 sq. units.

$$(1/2)|4 - k| \cdot 2 = 4 \Rightarrow k = 0 \text{ or } 8.$$

4. Solve by Cramer's rule: $2x - y = 5$, $x + 3y = -1$.

$$D = 7, D_x = 14, D_y = -7 \Rightarrow x = 2, y = -1.$$

Worksheet 5: Continuity and Differentiability

1. Differentiate $\sin(x^2 + 5)$.

$$2x \cos(x^2 + 5).$$

2. Differentiate x^x .

$$x^x(1 + \log x).$$

3. If $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$, find dy/dx .

$$\sin \theta / (1 - \cos \theta) = \cot(\theta/2).$$

4. Find k so that $f(x) = kx + 1$ ($x \leq \pi$), $\cos x$ ($x > \pi$) is continuous at $x = \pi$.

$$k\pi + 1 = -1 \Rightarrow k = -2/\pi.$$

Worksheet 6: Applications of Derivatives

1. Find the tangent to $y = x^2 - 2x + 7$ parallel to $2x - y + 9 = 0$.

$$2x - 2 = 2 \Rightarrow x = 2, y = 7. \text{ Tangent: } y = 2x + 3.$$

2. Find the local maxima and minima of $f(x) = x^3 - 6x^2 + 9x + 15$.

$$f' = 3(x - 1)(x - 3). \text{ Local max } 19 \text{ at } x = 1; \text{ local min } 15 \text{ at } x = 3.$$

3. Air is pumped into a spherical balloon at $900 \text{ cm}^3/\text{s}$. How fast does the radius grow when $r = 15 \text{ cm}$?

$$900 = 4\pi(225) dr/dt \Rightarrow dr/dt = 1/\pi \text{ cm/s.}$$

4. Find the absolute maximum and minimum of $f(x) = 2x^3 - 15x^2 + 36x + 1$ on $[1, 5]$.

$$f(1) = 24, f(2) = 29, f(3) = 28, f(5) = 56. \text{ Max } 56 \text{ at } x = 5; \text{ min } 24 \text{ at } x = 1.$$

Worksheet 7: Integrals

1. $\int (x^3 - 1)/x^2 dx$

$$x^2/2 + 1/x + C.$$

2. $\int \sin^3 x dx$

$$-\cos x + \cos^3 x/3 + C.$$

3. $\int dx/(x^2 - 4)$

$$(1/4) \log |(x - 2)/(x + 2)| + C.$$

4. $\int_0^{\pi/4} \tan^2 x dx$

$$[\tan x - x]_0^{\pi/4} = 1 - \pi/4.$$

5. $\int x \log x dx$

$$(x^2/2) \log x - x^2/4 + C.$$

6. $\int 2x/[(x + 1)(x + 2)] dx$

$$-2 \log|x + 1| + 4 \log|x + 2| + C.$$

Worksheet 8: Applications of Integrals

1. Find the area under $y = \sqrt{x}$ from $x = 0$ to $x = 4$.

$$(2/3)x^{3/2} \text{ from } 0 \text{ to } 4 = 16/3 \text{ sq. units.}$$

2. Find the area bounded by $y = x^2$ and the line $y = 4$.

$$\int_{-2}^2 (4 - x^2) dx = 32/3 \text{ sq. units.}$$

Worksheet 9: Differential Equations

1. Solve $dy/dx = (1 + y^2)/(1 + x^2)$.

$$\tan^{-1} y = \tan^{-1} x + C.$$

2. Solve $dy/dx + y = e^{-x}$.

$$\text{IF} = e^x: y e^x = x + C.$$

3. Solve $x dy/dx - y = x^2$ ($x > 0$).

$$dy/dx - y/x = x; \text{IF} = 1/x: y/x = x + C \Rightarrow y = x^2 + Cx.$$

4. Find the order and degree of $d^2y/dx^2 = [1 + (dy/dx)^2]^{3/2}$.

Square both sides to remove the fractional power: order 2, degree 2.

Worksheet 10: Vectors

1. For $\mathbf{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\mathbf{b} = 3\hat{i} + 2\hat{j} + \hat{k}$, find $\mathbf{a} \cdot \mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$.

$$\mathbf{a} \cdot \mathbf{b} = 10; \mathbf{a} \times \mathbf{b} = -4\hat{i} + 8\hat{j} - 4\hat{k}.$$

2. Find the angle between the vectors of question 1.

$$|\mathbf{a}| = |\mathbf{b}| = \sqrt{14}; \cos \theta = 10/14 = 5/7; \theta = \cos^{-1}(5/7).$$

3. Find the area of the triangle with vertices $\mathbf{A}(1, 1, 1)$, $\mathbf{B}(1, 2, 3)$, $\mathbf{C}(2, 3, 1)$.

$$\mathbf{AB} \times \mathbf{AC} = (0, 1, 2) \times (1, 2, 0) = (-4, 2, -1); \text{area} = \sqrt{21}/2 \text{ sq. units.}$$

4. Find λ if $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} - 3\hat{k}$ and $3\hat{i} + \lambda\hat{j} + 5\hat{k}$ are coplanar.

$$\text{Scalar triple product } 7\lambda + 28 = 0 \Rightarrow \lambda = -4.$$

Worksheet 11: Three-dimensional Geometry

1. Write the equation of the line through $(1, 2, 3)$ parallel to $3\hat{i} + 2\hat{j} - 2\hat{k}$.

$$(x - 1)/3 = (y - 2)/2 = (z - 3)/(-2).$$

2. Find the angle between the planes $x + y + 2z = 9$ and $2x - y + z = 15$.

$$\cos \theta = (2 - 1 + 2)/(\sqrt{6} \cdot \sqrt{6}) = 1/2 \Rightarrow \theta = 60^\circ.$$

3. Find the plane through $(1, 1, 0)$, $(1, 2, 1)$ and $(-2, 2, -1)$.

$$\text{Normal} = (0, 1, 1) \times (-3, 1, -1) = (-2, -3, 3). \text{Plane: } 2x + 3y - 3z = 5.$$

4. Find the distance of $(3, -2, 1)$ from $2x - y + 2z + 3 = 0$.

$$|6 + 2 + 2 + 3|/3 = 13/3 \text{ units.}$$

Worksheet 12: Linear Programming

1. Maximise $Z = 3x + 4y$ subject to $x + y \leq 4$, $x, y \geq 0$.

$$\text{Corners } (0, 0), (4, 0), (0, 4): Z = 0, 12, 16. \text{Max } 16 \text{ at } (0, 4).$$

2. Minimise $Z = 200x + 500y$ subject to $x + 2y \geq 10$, $3x + 4y \leq 24$, $x, y \geq 0$.

$$\text{Corners } (0, 5), (4, 3), (0, 6): Z = 2500, 2300, 3000. \text{Min } 2300 \text{ at } (4, 3).$$

Worksheet 13: Probability

1. $P(A) = 0.8$, $P(B) = 0.5$, $P(B | A) = 0.4$. Find $P(A \cap B)$, $P(A | B)$ and $P(A \cup B)$.

$$P(A \cap B) = 0.32; P(A | B) = 0.64; P(A \cup B) = 0.8 + 0.5 - 0.32 = 0.98.$$

2. A bag has 4 red and 6 black balls. Two are drawn without replacement. Find $P(\text{both red})$.

$$(4/10)(3/9) = 2/15.$$

3. $X \sim B(5, 1/5)$. Find $P(X \geq 1)$.

$$1 - (4/5)^5 = 1 - 1024/3125 = 2101/3125.$$

4. Find the mean and variance of the number shown on a fair die.

$$\text{Mean} = 21/6 = 7/2; \Sigma x^2 p = 91/6; \text{variance} = 91/6 - 49/4 = 35/12.$$

FINAL REMINDERS

Write units for areas, rates and volumes. Show the corner-point table in LPP. In probability, define events before using Bayes' theorem. Recheck each answer by substitution where possible.

PRACTICE & REVISION

Mathematics · +2 2nd Year (Class XII)

Mathematics – Extra Solved Problems

1. Show that the relation R on R defined by $R = \{(a, b) : a \leq b^2\}$ is neither reflexive, symmetric nor transitive.

Not reflexive: $(1/2, 1/2)$: $1/2 \leq 1/4$ is false. Not symmetric: $(1, 4) \in R$ ($1 \leq 16$) but $(4, 1) \notin R$ ($4 \leq 1$ false). Not transitive: $(3, 2) \in R$ ($3 \leq 4$), $(2, 1.5) \in R$ ($2 \leq 2.25$) but $(3, 1.5) \notin R$ ($3 \leq 2.25$ false).

2. Prove that $\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4$.

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}[(x+y)/(1-xy)]$ ($xy < 1$) $= \tan^{-1}[(5/6)/(5/6)] = \tan^{-1}1 = \pi/4$.

3. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, show that $A^2 - 4A + I = O$ and hence find A^{-1} .

$A^2 = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$. $4A = \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix}$. $A^2 - 4A + I = \begin{bmatrix} 7-8+1 & 0 \\ 0 & 7-8+1 \end{bmatrix} = O$ ✓. Multiply by A^{-1} : $A - 4I + A^{-1} = O \Rightarrow A^{-1} = 4I - A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

4. Using properties, prove $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

$R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$: $\begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$. Take $(a-b)$ and $(b-c)$ common: $(a-b)(b-c) \begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$. Expand along C_1 : $1 \times [(b+c) - (a+b)] = c-a$. Result $(a-b)(b-c)(c-a)$.

5. Find dy/dx if $y = (\log x)^{\cos x}$.

$\log y = \cos x \cdot \log(\log x)$. $y'/y = -\sin x \cdot \log(\log x) + \cos x/(x \log x)$. $dy/dx = (\log x)^{\cos x} [\cos x/(x \log x) - \sin x \log(\log x)]$.

6. Find the equation of the tangent to $y = x^2 - 2x + 7$ parallel to $2x - y + 9 = 0$.

Slope required 2. $dy/dx = 2x - 2 = 2 \Rightarrow x = 2, y = 7$. Tangent: $y - 7 = 2(x - 2) \Rightarrow \mathbf{2x - y + 3 = 0}$.

7. A rectangular sheet of tin 45 cm $\times$ 24 cm is made into an open box by cutting equal squares from corners. Find the side of the square for maximum volume.

$V = x(45 - 2x)(24 - 2x) = 4x^3 - 138x^2 + 1080x$. $V' = 12x^2 - 276x + 1080 = 12(x^2 - 23x + 90) = 12(x - 5)(x - 18) = 0$. $x = 18$ impossible ($24 - 36 < 0$). $V''(5) = 24(5) - 276 < 0 \Rightarrow \text{max. } \mathbf{x = 5 \text{ cm}}$.

8. Evaluate $\int (3x - 2)/[(x + 1)^2(x + 3)] dx$.

$(3x - 2)/[(x+1)^2(x+3)] = A/(x+1) + B/(x+1)^2 + C/(x+3)$. $x = -1: -5 = 2B \Rightarrow B = -5/2$. $x = -3: -11 = 4C \Rightarrow C = -11/4$. Compare x^2 : $0 = A + C \Rightarrow A = 11/4$. Integral = $(11/4)\log|x+1| + 5/(2(x+1)) - (11/4)\log|x+3| + C$.

9. Evaluate $\int e^x(1/x - 1/x^2) dx$.

$f(x) = 1/x, f'(x) = -1/x^2 \Rightarrow \mathbf{e^x/x + C}$.

10. Evaluate $\int_0^\pi x \sin x/(1 + \cos^2 x) dx$.

Using P4: $I = \int (\pi - x) \sin x/(1 + \cos^2 x) dx \Rightarrow 2I = \pi \int_0^\pi \sin x/(1 + \cos^2 x) dx$. Put $\cos x = t$: $\pi \int_{-1}^1 dt/(1 + t^2) = \pi(\pi/4 + \pi/4) = \pi^2/2 \Rightarrow \mathbf{I = \pi^2/4}$.

11. Find the area of the region enclosed by $y = x^2$ and $y = x$.

Intersect at $x = 0, 1$. Area = $\int_0^1 (x - x^2) dx = 1/2 - 1/3 = \mathbf{1/6 \text{ sq unit}}$.

12. Solve $(1 + x^2)dy/dx + 2xy = 1/(1 + x^2)$, $y = 0$ when $x = 1$.

$dy/dx + [2x/(1 + x^2)]y = 1/(1 + x^2)^2$. I.F. = $1 + x^2$. $y(1 + x^2) = \int dx/(1 + x^2) = \tan^{-1} x + C$. At $x = 1$: $0 = \pi/4 + C$. $\mathbf{y(1 + x^2) = \tan^{-1} x - \pi/4}$.

13. Show that the points $A(2\hat{i} - \hat{j} + \hat{k})$, $B(\hat{i} - 3\hat{j} - 5\hat{k})$, $C(3\hat{i} - 4\hat{j} - 4\hat{k})$ form a right-angled triangle.

$AB = -\hat{i} - 2\hat{j} - 6\hat{k}$; $BC = 2\hat{i} - \hat{j} + \hat{k}$; $CA = -\hat{i} + 3\hat{j} + 5\hat{k}$. $AB \cdot BC = -2 + 2 - 6 = -6 \neq 0$; $BC \cdot CA = -2 - 3 + 5 = 0 \Rightarrow \text{right angle at C}$.

14. Find a unit vector perpendicular to both $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

$\vec{a} \times \vec{b} = \hat{i}(3 - 2) - \hat{j}(3 - 1) + \hat{k}(2 - 1) = \hat{i} - 2\hat{j} + \hat{k}$; $|\cdot| = \sqrt{6}$. Unit vector $\mathbf{\pm(\hat{i} - 2\hat{j} + \hat{k})/\sqrt{6}}$.

15. Find the shortest distance between $(x + 1)/7 = (y + 1)/(-6) = (z + 1)/1$ and $(x - 3)/1 = (y - 5)/(-2) = (z - 7)/1$.

$\vec{a}_1 = (-1, -1, -1)$, $\vec{b}_1 = (7, -6, 1)$; $\vec{a}_2 = (3, 5, 7)$, $\vec{b}_2 = (1, -2, 1)$. $\vec{b}_1 \times \vec{b}_2 = (-6 + 2, -(7 - 1), -14 + 6) = (-4, -6, -8)$; magnitude $\sqrt{116} = 2\sqrt{29}$. $\vec{a}_2 - \vec{a}_1 = (4, 6, 8)$. Dot = $-16 - 36 - 64 = -116$. $d = 116/(2\sqrt{29}) = 2\sqrt{29}$.

16. A die is thrown twice; find the probability of getting a sum of 7 given that the first throw is an odd number.

F = first odd: 18 outcomes. $E \cap F: (1,6), (3,4), (5,2) = 3$. $P(E|F) = 3/18 = 1/6$.

17. A person tells the truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

$P(6) = 1/6$, $P(\text{not } 6) = 5/6$. $P(\text{report } 6 | 6) = 3/4$; $P(\text{report } 6 | \text{not } 6) = 1/4$. $P = (1/6 \times 3/4)/(1/6 \times 3/4 + 5/6 \times 1/4) = (3/24)/(8/24) = 3/8$.

18. Find the mean of the number of aces in drawing two cards successively with replacement from a pack of 52.

$p = 1/13$, $q = 12/13$. $P(0) = 144/169$, $P(1) = 24/169$, $P(2) = 1/169$. Mean = $24/169 + 2/169 = 26/169 = 2/13$.

Mathematics – Formula Sheet (Class XII)

Topic	Key formulas
Inverse trig	Ranges: $\sin^{-1} [-\pi/2, \pi/2]$; $\cos^{-1} [0, \pi]$; $\tan^{-1} (-\pi/2, \pi/2)$. $\sin^{-1} x + \cos^{-1} x = \pi/2$
Matrices	$(AB)' = B'A'$; $(AB)^{-1} = B^{-1}A^{-1}$; $A = \frac{1}{2}(A+A')$ + $\frac{1}{2}(A-A')$
Determinants	$A^{-1} = \text{adj } A/ A $; $ kA = k^n A $; $ \text{adj } A = A ^{n-1}$; area = $\frac{1}{2} \det $
Derivatives	$(\sin^{-1} x)' = 1/\sqrt{1-x^2}$; $(\tan^{-1} x)' = 1/(1+x^2)$; $(e^x)' = e^x$; $(\log x)' = 1/x$; parametric $dy/dx = (dy/dt)/(dx/dt)$
Applications	$f' > 0$ increasing; $f'(c) = 0$, $f''(c) < 0$ max, $f''(c) > 0$ min
Integrals	$\int dx/(x^2+a^2) = (1/a)\tan^{-1}(x/a)$; $\int dx/\sqrt{a^2-x^2} = \sin^{-1}(x/a)$; by parts ILATE; $\int e^x[f+f'] = e^x f$
Definite	$\int_0^a f(x) = \int_0^a f(a-x)$; odd $\rightarrow 0$; even $\rightarrow 2\int_0^a$
Area	Circle πa^2 ; ellipse πab ; $\int_a^b y dx$
D.E.	Homogeneous: $y = vx$; Linear: I.F. = $e^{\int P dx}$, $y \cdot \text{I.F.} = \int Q \cdot \text{I.F.} dx + C$
Vectors	$\vec{a} \cdot \vec{b} = \vec{a} \vec{b} \cos\theta$; $ \vec{a} \times \vec{b} = \vec{a} \vec{b} \sin\theta$; projection = $\vec{a} \cdot \vec{b}/ \vec{b} $; area of $\Delta = \frac{1}{2} \vec{a} \times \vec{b} $
3-D lines	$\vec{r} = \vec{a} + \lambda \vec{b}$; SD = $ (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) / \vec{b}_1 \times \vec{b}_2 $
LPP	Evaluate Z at corner points of feasible region
Probability	$P(E F) = P(E \cap F)/P(F)$; independent: $P(E \cap F) = P(E)P(F)$; Bayes; $E(X) = \sum x_i p_i$

Model Question Paper – Mathematics Class XII (with hints)

Time: 3 hours · Full marks: 80

Section A – 1 mark each

- Find the principal value of $\tan^{-1}(-1)$. $-\pi/4$
- If A is a square matrix with $|A| = 5$ of order 2, find $|3A|$. $9 \times 5 = 45$
- Differentiate $\sin(\log x)$. $\cos(\log x)/x$
- Evaluate $\int_{-1}^1 x^3 dx$. 0
- Find the order and degree of $(y'')^2 + y' = 0$. Order 2, degree 2
- Find $|\vec{a} \times \vec{b}|$ if $|\vec{a}| = 2$, $|\vec{b}| = 3$ and $\theta = 30^\circ$. 3
- If $P(A) = 0.4$, $P(B) = 0.5$ and A, B independent, find $P(A \cap B)$. 0.2
- Define the objective function in LPP. Linear function $Z = ax + by$ to be optimised

Section B – 2-4 marks each

- Show that $f(x) = 2x + 3$ is one-one and onto on R.
- Find k if $f(x) = kx^2$ ($x \leq 2$), 3 ($x > 2$) is continuous at $x = 2$. $k = 3/4$
- Find the intervals where $f(x) = x^2 - 4x + 6$ is increasing. $(2, \infty)$
- Evaluate $\int x \log x dx$. $(x^2/2)\log x - x^2/4 + C$
- Solve $dy/dx = (1 + x)(1 + y^2)$. $\tan^{-1}y = x + x^2/2 + C$
- Find the angle between the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$. $\cos \theta = 10/14 = 5/7$

Section C – 6 marks each

- Solve by matrix method: $3x - 2y + 3z = 8$, $2x + y - z = 1$, $4x - 3y + 2z = 4$. Ans: $x = 1$, $y = 2$, $z = 3$
- Find the area of the ellipse $x^2/16 + y^2/9 = 1$. 12π
- Bag A: 2 white, 3 red; bag B: 4 white, 5 red. A ball drawn at random from a randomly chosen bag is red. Find the probability it came from bag B. $(\frac{1}{2} \cdot 5/9) / (\frac{1}{2} \cdot 3/5 + \frac{1}{2} \cdot 5/9) = 25/52$

Mathematics – More Integration and Determinant Practice

1. $\int dx/(1 + \tan x)$

$$= \int \cos x / (\sin x + \cos x) dx = \frac{1}{2} \int [(\cos x + \sin x) + (\cos x - \sin x)] / (\sin x + \cos x) dx = x/2 + \frac{1}{2} \log|\sin x + \cos x| + C.$$

2. $\int \sin^3 x \cos^2 x dx$

$$= \int (1 - \cos^2 x) \cos^2 x \sin x dx; \cos x = t, -dt = \sin x dx: -\int (t^2 - t^4) dt = -t^3/3 + t^5/5 = -\cos^3 x/3 + \cos^5 x/5 + C.$$

3. $\int x \tan^{-1} x \, dx$

By parts ($u = \tan^{-1} x$): $(x^2/2)\tan^{-1} x - \frac{1}{2}\int x^2/(1+x^2)dx = (x^2/2)\tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x) + C$.

4. $\int e^x \sin x \, dx$

$I = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - [e^x \cos x + \int e^x \sin x \, dx] \Rightarrow 2I = e^x(\sin x - \cos x) \Rightarrow I = (e^x/2)(\sin x - \cos x) + C$.

5. $\int \sqrt{x^2 + 4x + 6} \, dx$

$= \int \sqrt{(x+2)^2 + 2} \, dx = ((x+2)/2)\sqrt{x^2 + 4x + 6} + \log|x+2 + \sqrt{x^2 + 4x + 6}| + C$.

6. $\int_0^{\pi/4} \log(1 + \tan x) \, dx$

Using P5: $1 + \tan(\pi/4 - x) = 2/(1 + \tan x)$. So $I = \int [\log 2 - \log(1 + \tan x)]dx = (\pi/4)\log 2 - I \Rightarrow I = (\pi/8) \log 2$.

7. $\int_1^4 |x - 2| \, dx$

$= \int_1^2 (2-x)dx + \int_2^4 (x-2)dx = 1/2 + 2 = 5/2$.

8. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, find A^{-1} and solve $x - y + z = 4$, $2x + y - 3z = 0$, $x + y + z = 2$.

$|A| = 1(1+3) + 1(2+3) + 1(2-1) = 4+5+1 = 10$. Cofactors: $A_{11} = 4$, $A_{12} = -5$, $A_{13} = 1$; $A_{21} = 2$, $A_{22} = 0$, $A_{23} = -2$; $A_{31} = 2$, $A_{32} = 5$, $A_{33} = 3$. $\text{adj } A = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$. $X = (1/10)[16+0+4; -20+0+10; 4-0+6] = (1/10)[20; -10; 10] \Rightarrow x = 2, y = -1, z = 1$.

9. If $x = a \cos^3 \theta$, $y = a \sin^3 \theta$, find dy/dx .

$dx/d\theta = -3a \cos^2 \theta \sin \theta$; $dy/d\theta = 3a \sin^2 \theta \cos \theta$. $dy/dx = -\tan \theta$.

10. A ladder 5 m long leans against a wall. Its bottom is pulled away at 2 cm/s. How fast is its top sliding down when the bottom is 4 m from the wall?

$x^2 + y^2 = 25 \Rightarrow 2x \, dx/dt + 2y \, dy/dt = 0$. At $x = 4$, $y = 3$: $dy/dt = -(4/3)(2) = -8/3 \text{ cm/s}$ (sliding down at $8/3 \text{ cm/s}$).

Last-Minute Exam Tips

Mathematics

- Never forget "+ C" in indefinite integrals — examiners deduct marks.
- In definite integrals, check whether the function is odd/even or whether P4/P5 can be used before integrating directly.

- For principal values, always check the answer lies in the correct range (e.g. $\cos^{-1}$ in $[0, \pi]$).
- In LPP, draw a neat graph, shade the feasible region, list corner points in a table and write the conclusion in words.
- In Bayes' theorem problems, first define the events E_1 , E_2 , A in words — this earns step marks.
- For matrix-method questions, show $|A|$, all cofactors, $\text{adj } A$, A^{-1} and then $X = A^{-1}B$; finally verify by substitution.
- For maxima/minima word problems, write the function in one variable, then first and second derivative tests, and state the answer with units.

THE NIGHT BEFORE THE EXAM

Revise only formula boxes, name reactions, grammar rules and your own mistake list. Sleep well — a fresh mind solves problems faster than a tired one. Carry admit card, two pens, pencil, scale, eraser and a geometry box. Good luck!